A complete deductive calculus for (implications of) coequations

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Outline

I. Preliminaries

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II. Quasi-covarieties and covarieties

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IV. The Covariety Theorems

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Coalgebras

Given a functor $\Gamma : \mathcal{C} \to \mathcal{C}$, a Γ-coalgebra is a pair

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where $C \in \mathcal{C}$ and $\gamma : C \to \Gamma C$.

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The category of Γ-coalgebras and their homomorphisms is denoted $\mathcal{C}_\Gamma$.

Factorization systems

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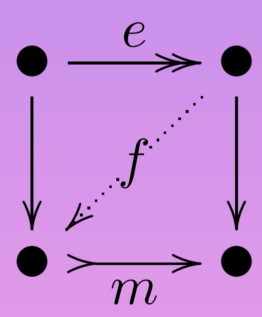
- $\mathcal{H}$ and $\mathcal{S}$ satisfy the *diagonal fill-in property* , namely, for every commutative square



where $e \in \mathcal{H}$ and $m \in \mathcal{S}$, there is a unique arrow f, as shown, making each triangle commute ;

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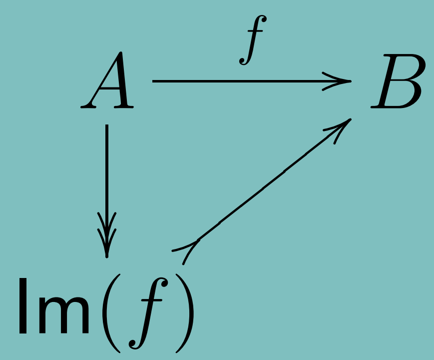
If $\mathcal{C}$ has a factorization system, then any arrow $f : A \to B$ can be factored uniquely up to isomorphism thus.



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For each $C \in \mathcal{C}$, define

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$\mathcal{C}$ is $\mathcal{S}$-*well-powered* if, for every $C \in \mathcal{C}$, $\mathsf{Sub}(C)$ is a set.

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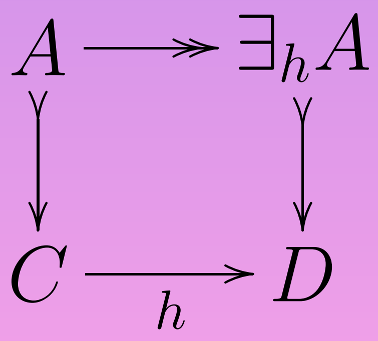
Each $h\colon C \to D$ induces a morphism $\exists_h\colon \mathsf{Sub}(C) \to \mathsf{Sub}(D)$

by $\exists_h(A \overset{i}{\rightarrowtail} C) = \mathsf{Im}(i \circ h)$.



Factorization systems for coalgebras

Let $\langle \mathcal{H}, \mathcal{S} \rangle$ be a factorization system and suppose that $\Gamma : \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{S}$-morphisms (i.e., if $i \in \mathcal{S}$, then $\Gamma i \in \mathcal{S}$) .

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Let $\langle \mathcal{H}, \mathcal{S} \rangle$ be a factorization system and suppose that $\Gamma : \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{S}$-morphisms.

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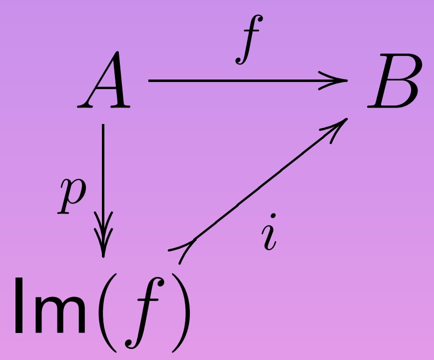
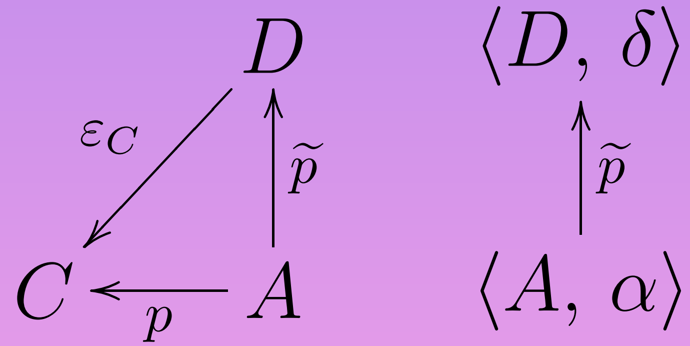
In other words, every Γ-homomorphism $f : \langle A, \alpha \rangle \to \langle B, \beta \rangle$ factors uniquely as in



where p and i are Γ-homomorphisms.

Cofree coalgebras

Let $\langle D, \delta\rangle$ be given, together with a C-coloring $\varepsilon_C : D \to C$ of D.
We say that $\langle D, \delta\rangle$ is cofree over C just in case, for every coalgebra $\langle A, \alpha\rangle$ and every coloring $p : A \to C$, there is a unique homomorphism $\widetilde{p} : \langle A, \alpha\rangle \to \langle D, \delta\rangle$ such that the diagram below commutes.



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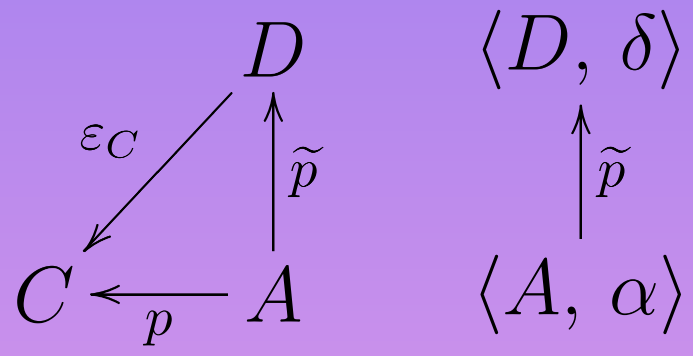

For any coloring $p : A \to C$, there is a Γ-homomorphism $\widetilde{p} : \langle A,\ \alpha\rangle \to \langle D,\ \delta\rangle$ “consistent” with p.

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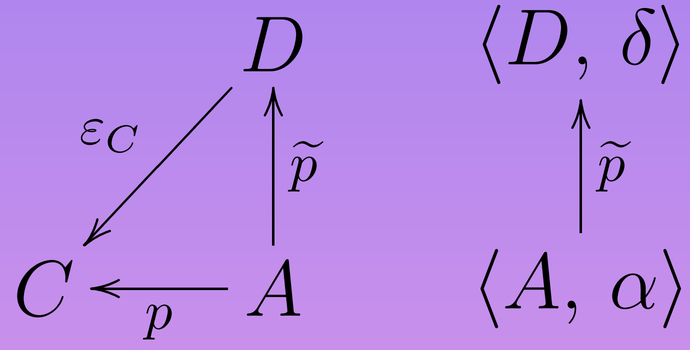
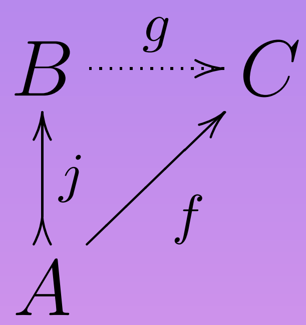


If, for every object $C \in \mathcal{C}$, there is a cofree $\langle D,\ \delta\rangle$ over C, then we have an adjunction

$$\mathcal{C}_\Gamma \underset{H}{\overset{U}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}.$$

$\mathcal{S}$-injectives

An object $C \in \mathcal{C}$ is *$\mathcal{S}$-injective* if, for all $j : A \rightarrowtail B$ in $\mathcal{S}$, and all $f : A \to C$, there is a (not necessarily unique) extension $g : B \to C$ making the diagram below commute.



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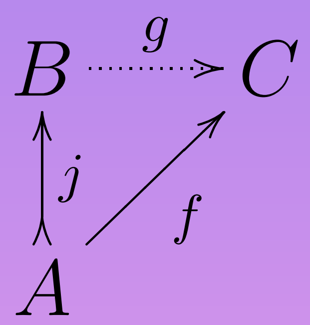

$\mathcal{C}$ *has enough $\mathcal{S}$-injectives* iff for every $A \in \mathcal{C}$, there is an $\mathcal{S}$-injective $C \in \mathcal{C}$ and a $\mathcal{S}$-morphism $A \rightarrowtail C$.

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Theorem. *If* $U : \mathcal{C}_\Gamma \to \mathcal{C}$ *has a right adjoint* H *and* $\mathcal{C}$ *has enough* $\mathcal{S}$*-injectives, then* $\mathcal{C}_\Gamma$ *has enough* $U^{-1}\mathcal{S}$*-injectives.*

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Proof. Let $\langle A, \alpha \rangle$ be given and $A \leq C$, where C is $\mathcal{S}$-injective. Then $\langle A, \alpha \rangle \leq HC$. It suffices to show HC is $U^{-1}\mathcal{S}$-injective.

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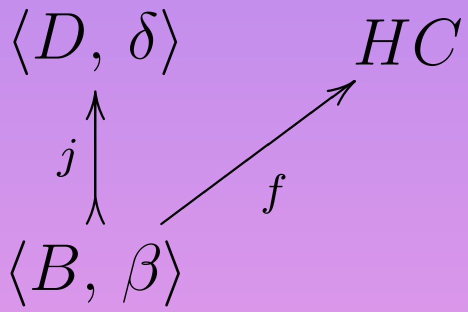

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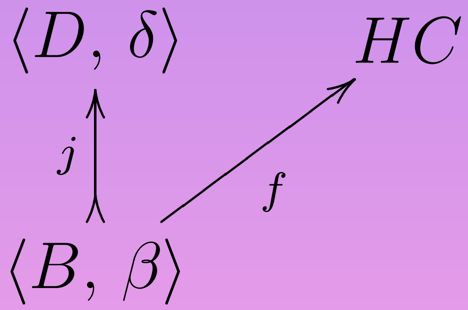

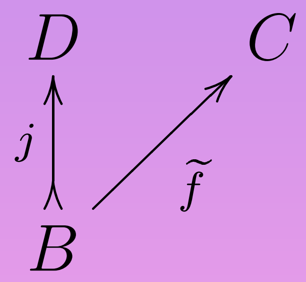

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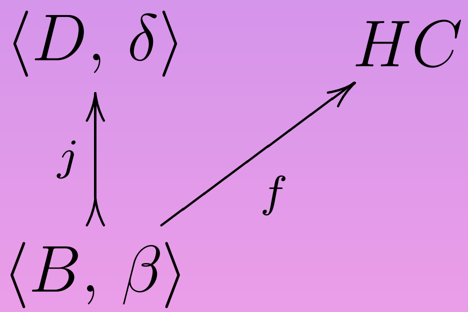

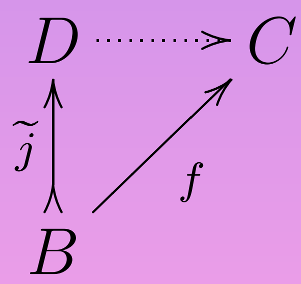

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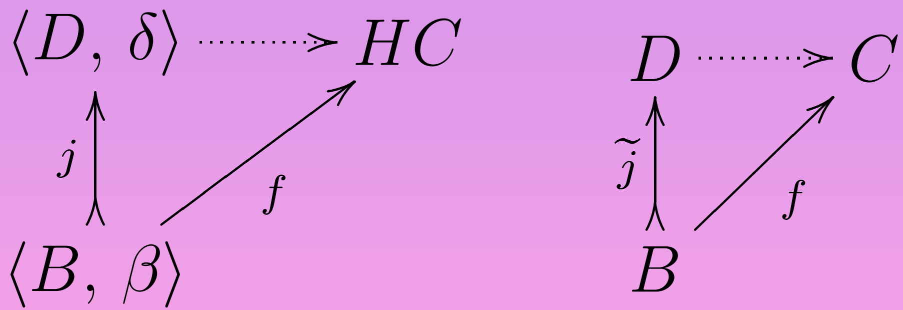
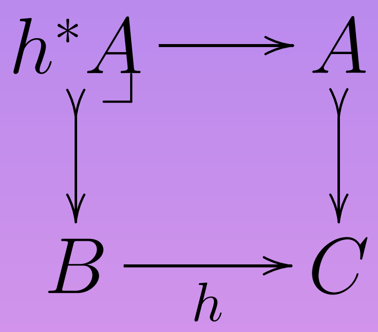


About $\mathcal{S}$-meets

Recall that $\mathsf{Sub}(C)$ denotes the poset of isomorphism classes of $\mathcal{S}$-morphisms into C.
In any factorization system $\langle \mathcal{H}, \mathcal{S} \rangle$, the $\mathcal{S}$-morphisms are stable under pullbacks.



Thus, if $\mathcal{C}$ has pullbacks of $\mathcal{S}$-morphisms, then each $h : B \to C$ induces a functor $h^* : \mathsf{Sub}(C) \to \mathsf{Sub}(B)$.

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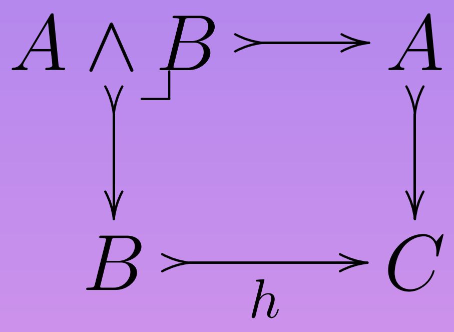

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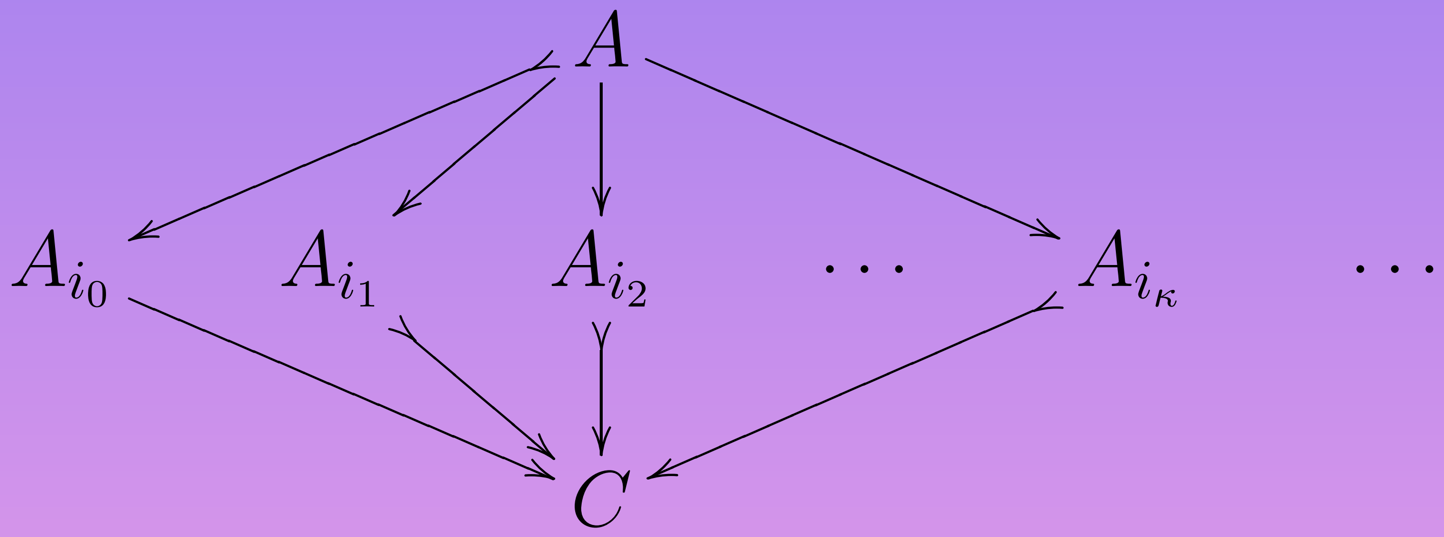
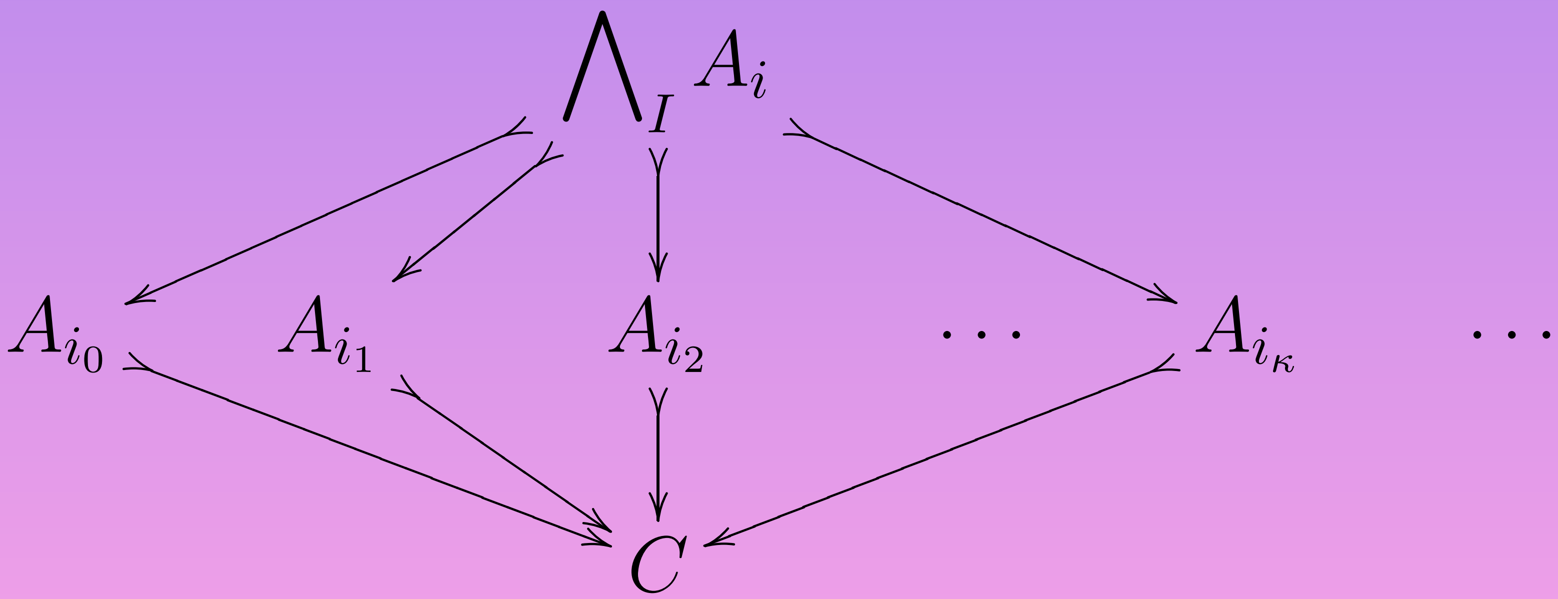


About $\mathcal{S}$-meets

In any factorization system $\langle \mathcal{H}, \mathcal{S} \rangle$, the $\mathcal{S}$-morphisms are stable under generalized pullbacks. Assuming that $\mathcal{C}$ has such limits, this gives one a notion of $\bigwedge_I$ for $\mathsf{Sub}(C)$,

$$\bigwedge_I : \mathsf{Sub}(C)^I \to \mathsf{Sub}(C).$$



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- If $\mathcal{C}$ is $\mathcal{S}$-well-powered, then $\mathcal{C}_\Gamma$ is $U^{-1}\mathcal{S}$-well-powered.

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- If $\mathcal{C}$ is $\mathcal{S}$-well-powered, then $\mathcal{C}_\Gamma$ is $U^{-1}\mathcal{S}$-well-powered.
- If $\mathcal{C}$ has enough $\mathcal{S}$-injectives and $U : \mathcal{C}_\Gamma \to \mathcal{C}$ has a right adjoint, then $\mathcal{C}_\Gamma$ has enough $U^{-1}\mathcal{S}$-injectives.

Structural summary

Hereafter, we assume that $\mathcal{C}$ has all coproducts, a factorization system $\langle \mathcal{H}, \mathcal{S} \rangle$, enough $\mathcal{S}$-injectives and meets of $\mathcal{S}$-morphisms and is $\mathcal{S}$-well-powered, and that Γ-preserves $\mathcal{S}$-morphisms. We further assume that $U : \mathcal{C}_\Gamma \to \mathcal{C}$ has a right adjoint H.

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Quasi-covarieties and covarieties

Let $\mathbf{V} \subseteq \mathcal{C}_\Gamma$. We define

$$\mathcal{H}\mathbf{V} = \{\langle B,\, \beta\rangle \mid \exists \mathbf{V} \ni \langle C,\, \gamma\rangle \longrightarrow\!\!\!\!\!\to \langle B,\, \beta\rangle\}$$

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A coequational language

Fix a $\mathcal{S}$-injective $C \in \mathcal{C}$. We define a simple language $\mathcal{L}_{\mathsf{Coeq}}$ (properly, $\mathcal{L}^{C}_{\mathsf{Coeq}}$).

- For every P in $\mathsf{Sub}(UHC)$, we introduce an atomic proposition P in $\mathcal{L}_{\mathsf{Coeq}}$, i.e., $\mathsf{Sub}(UHC) \subseteq \mathcal{L}_{\mathsf{Coeq}}$.

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- If $\varphi \in \mathcal{L}_{\mathsf{Coeq}}$ and $h\colon HC \to HC$, then $\varphi(h(x)) \in \mathcal{L}_{\mathsf{Coeq}}$.

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A coequational language

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(Definition of $\Box$ forthcoming!)

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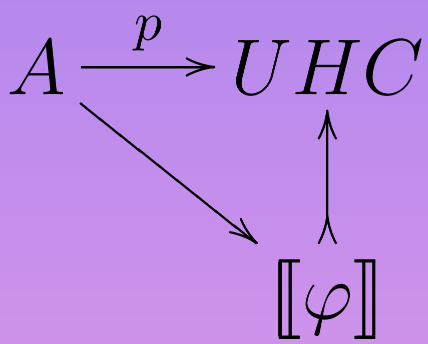
$$[\![\exists y(\varphi(y) \wedge h(y) = x)]\!] = \exists_h [\![\varphi]\!]$$

Coequations

A coalgebra $\langle A,\, \alpha\rangle$ satisfies φ iff for every homomorphism $p\colon\langle A,\, \alpha\rangle\to HC$, we have $\mathsf{Im}(p) \leq [\![\varphi]\!]$.

Coequations

A coalgebra $\langle A, \alpha\rangle$ satisfies φ iff for every homomorphism $p : \langle A, \alpha\rangle \to HC$, we have $\mathsf{Im}(p) \leq [\![\varphi]\!]$. In other words, $\langle A, \alpha\rangle \models \varphi$ iff every $p : \langle A, \alpha\rangle \to HC$ factors through $[\![\varphi]\!]$.



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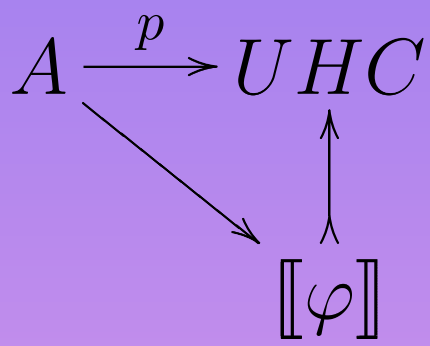
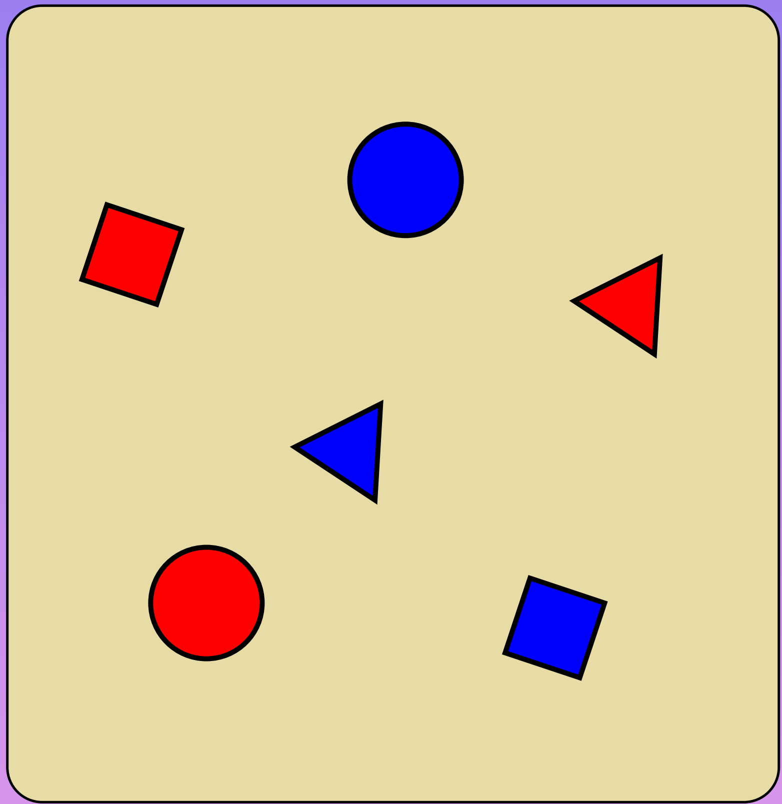


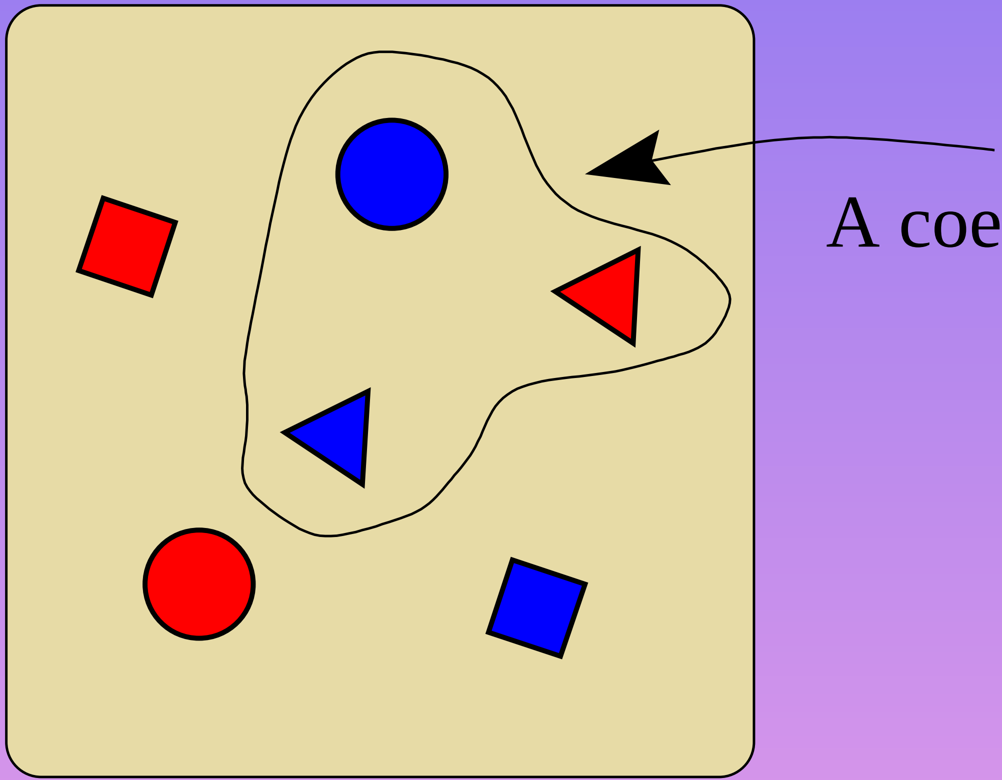
Homomorphisms $p\colon\langle A, \alpha\rangle \to HC$ correspond to colorings $\widetilde{p}\colon A \to C$. Thus, $\langle A, \alpha \rangle \models \varphi$ just in case, however we color A (via $\widetilde{p}$), the image of the corresponding homomorphism p lies in φ.

Example



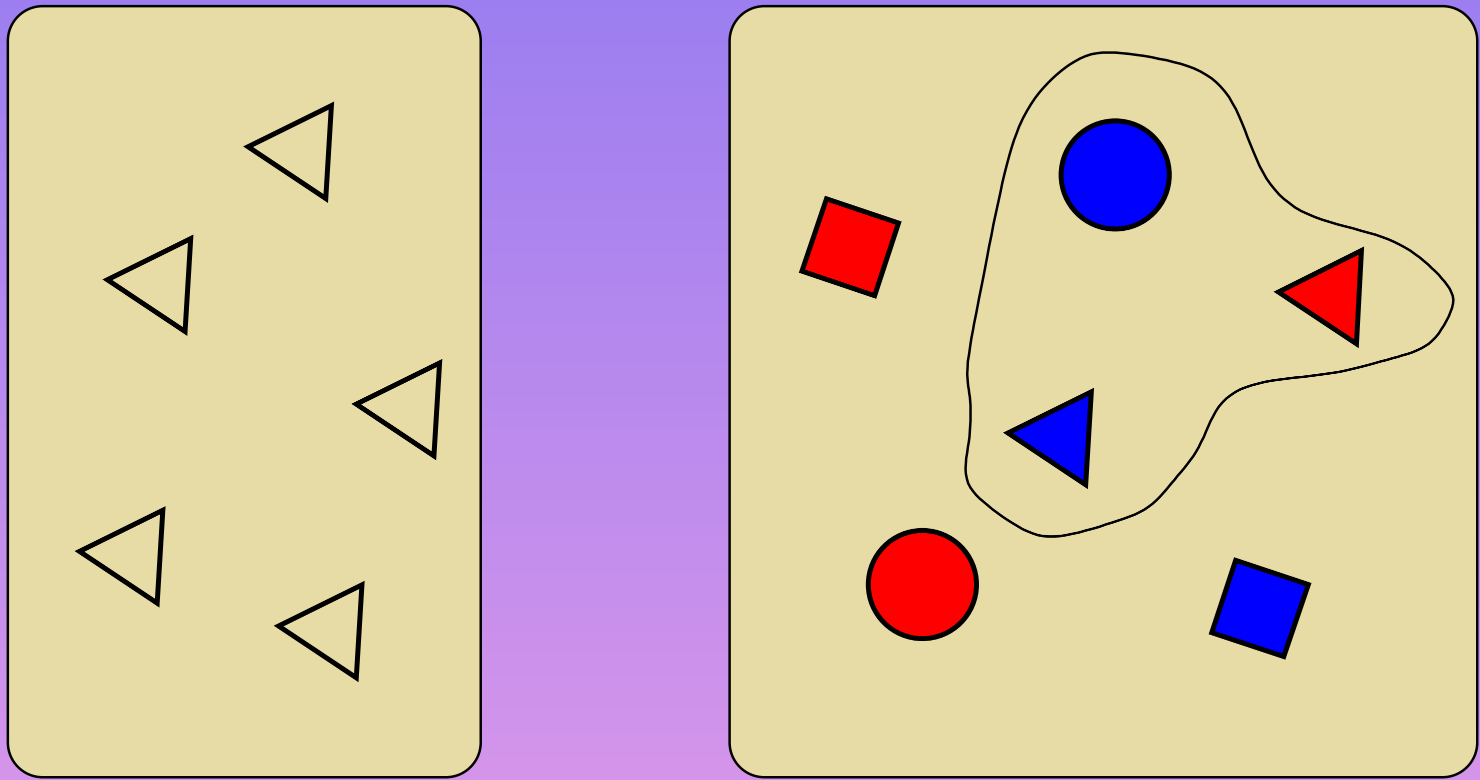
The cofree coalgebra $H2$

Example



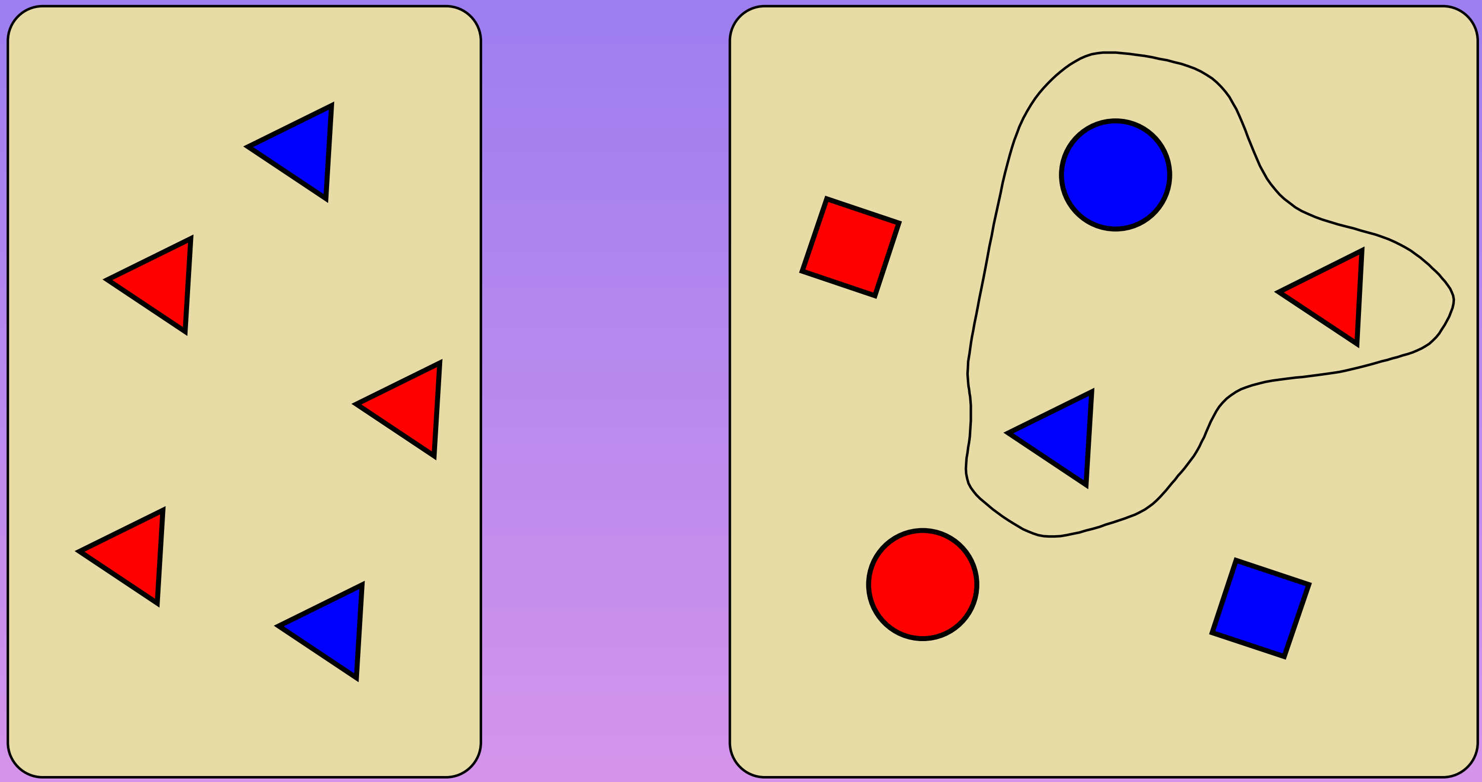
A coequation.

Example



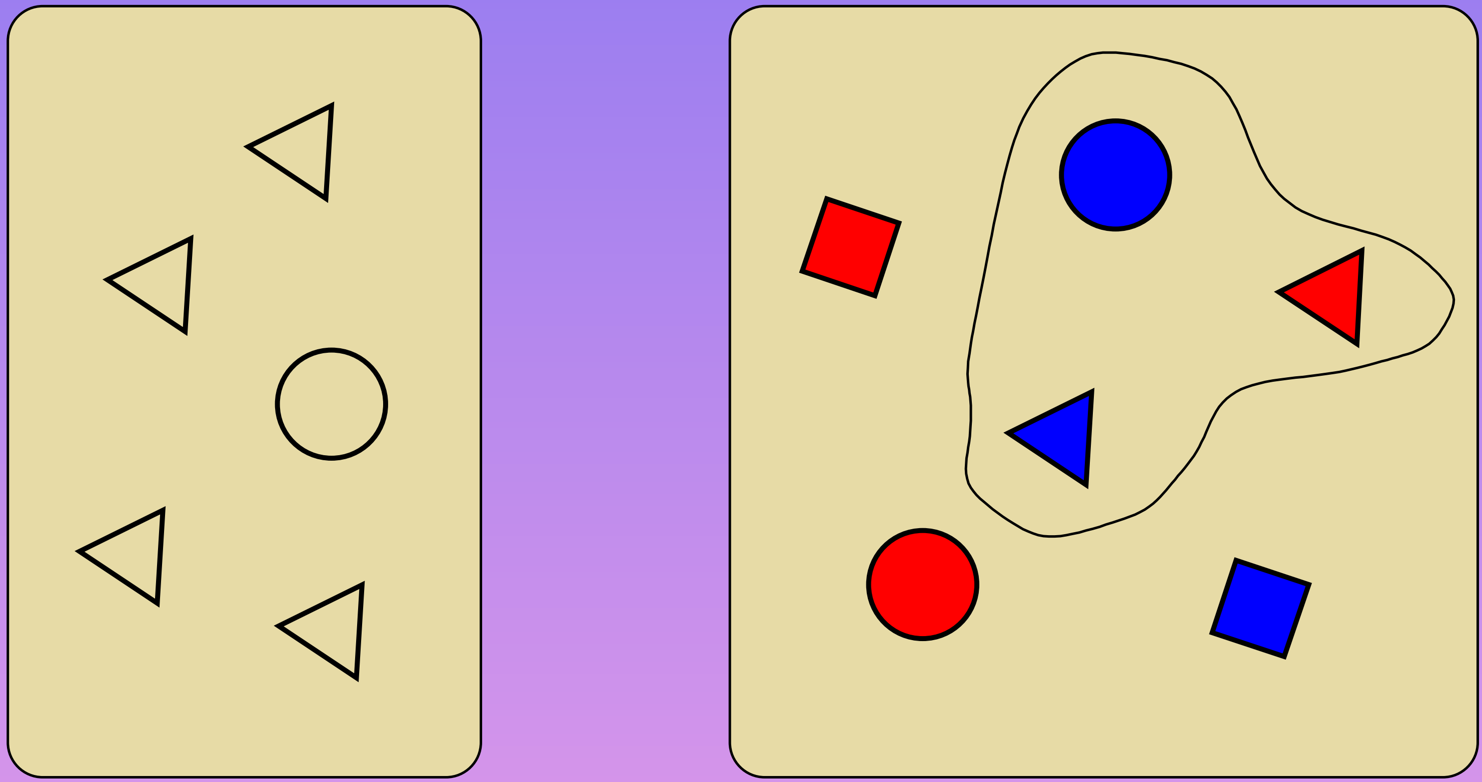
This coalgebra satisfies P.

Example



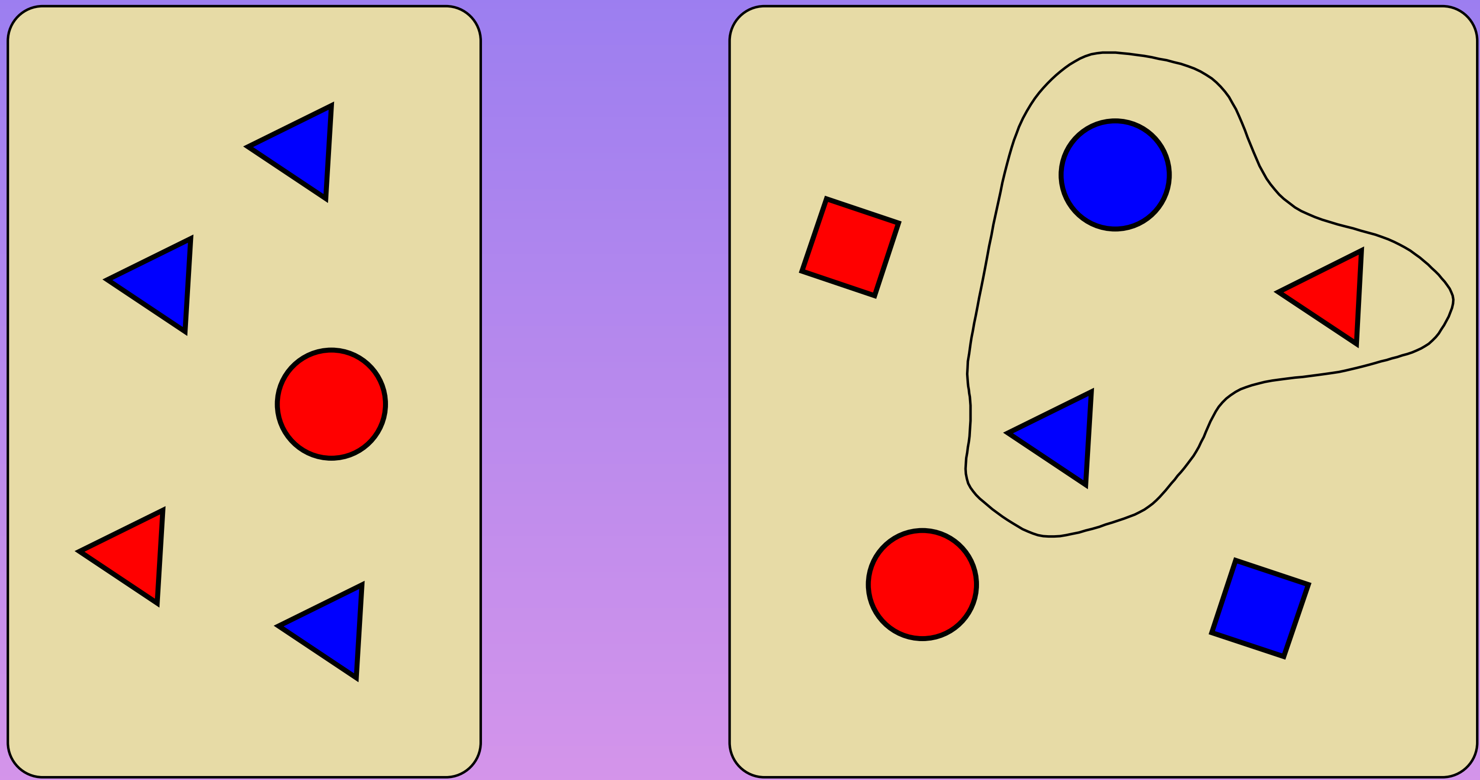
Under any coloring, the elements of the coalgebra map to elements of P.

Example



This coalgebra doesn't satisfy P.

Example



If we paint the circle red, it isn't mapped to an element of P.

An implicational language

Define $\mathcal{L}_{\mathsf{Imp}} = \{\varphi \Rightarrow \psi \mid \varphi, \psi \in \mathcal{L}_{\mathsf{Coeq}}\}$.
Say that $\langle A,\, \alpha\rangle \models \varphi \Rightarrow \psi$ just in case, for every $p{:}\langle A,\, \alpha\rangle{\to}HC$ such that $\mathsf{Im}(p) \leq [\![\varphi]\!]$, also $\mathsf{Im}(p) \leq [\![\psi]\!]$.

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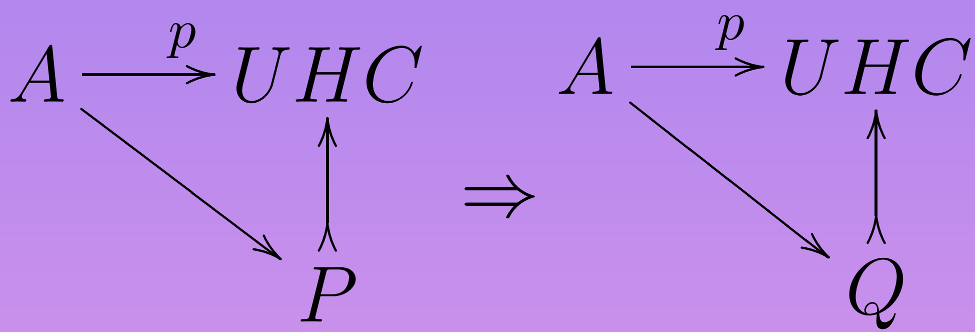

This is not the same as ($\langle A, \alpha\rangle \not\models \varphi$ or $\langle A, \alpha\rangle \models \psi$). That would be true if either there is some p such that $\mathsf{Im}(p) \not\leq [\![\varphi]\!]$ or for all p, $\mathsf{Im}(p) \leq [\![\psi]\!]$.

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Define $\mathcal{L}_{\mathsf{Imp}} = \{\varphi \Rightarrow \psi \mid \varphi, \psi \in \mathcal{L}_{\mathsf{Coeq}}\}$.
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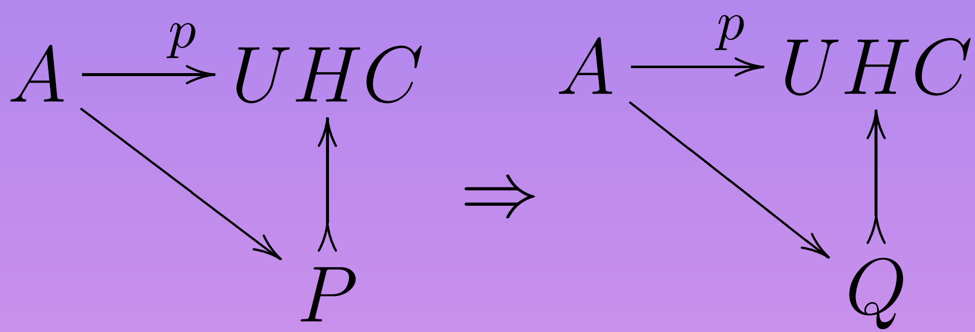

This is also not the same as $\langle A,\, \alpha\rangle \models \neg\varphi \vee \psi$ (if $\mathsf{Sub}(UHC)$ is a Heyting algebra).

An implicational language

Define $\mathcal{L}_{\mathsf{Imp}} = \{\varphi \Rightarrow \psi \mid \varphi, \psi \in \mathcal{L}_{\mathsf{Coeq}}\}$.
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$$\begin{array}{ccc} A & \xrightarrow{p} & UHC \\ & \searrow & \uparrow \\ & & P \end{array} \quad \Rightarrow \quad \begin{array}{ccc} A & \xrightarrow{p} & UHC \\ & \searrow & \uparrow \\ & & Q \end{array}$$

Note:

$$\langle A,\, \alpha\rangle \models \varphi \text{ iff } \langle A,\, \alpha\rangle \models \top \Rightarrow \varphi,$$

where $\top = (HC{=}HC)$.

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V. The Invariance Theorem
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VII. Coequational logic (Completeness)
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The Covariety Theorems

Given a class $\mathbf{V}$ of coalgebras, define

$$\mathsf{Th}(\mathbf{V}) = \{\varphi \in \mathcal{L}^{C}_{\mathsf{Coeq}} \mid \mathbf{V} \models \varphi,\ C\ \mathcal{S}\text{-injective}\},$$

$$\mathsf{Imp}(\mathbf{V}) = \{P \Rightarrow Q \in \mathcal{L}^{C}_{\mathsf{Imp}} \mid \mathbf{V} \models P \Rightarrow Q,\ P, Q \leq UHC,\ C\ \mathcal{S}\text{-injective}\}.$$

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Given a collection S of (implications between) coequations, define

$$\mathsf{Mod}(S) = \{\langle A,\, \alpha\rangle \mid \langle A,\, \alpha\rangle \models S\}.$$

The Covariety Theorems

Theorem (The "co-Birkhoff" theorem). *For any* $\mathbf{V}$*,*

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Theorem (The co-quasivariety theorem). *For any* $\mathbf{V}$,

$$\mathcal{H}\Sigma\mathbf{V} = \mathsf{Mod}\,\mathsf{Imp}(\mathbf{V}).$$

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Birkhoff's completeness theorem

So much for the formal dual of the variety theorem. What about the formal dual of Birkhoff's completeness theorem?

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Let S be a set of equations for an algebraic signature Σ. Let $\mathsf{Ded}(S)$ denote the deductive closure of S under the usual equational logic.

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Here, $\mathsf{Th}(\mathbf{V})$ denotes the equational theory of a class of algebras.

Birkhoff's completeness theorem

Theorem (Birkhoff's completeness theorem).
For any set S of equations,

$$\mathsf{Ded}(S) = \mathsf{Th}\,\mathsf{Mod}(S).$$

Compare this to the variety theorem, namely for every $\mathbf{V}$,

$$\mathcal{HSP}\mathbf{V} = \mathsf{Mod}\,\mathsf{Th}(\mathbf{V}).$$

Birkhoff's completeness theorem

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Main goal Find a logic on sets of coequations such that for any set S of coequations over C,

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First step Find the formal dual to Birkhoff's completeness theorem.

The invariance theorem

Define interior operators

$$\Box, \boxslash : \mathsf{Sub}(UHC) \longrightarrow \mathsf{Sub}(UHC)$$

by

$$\Box P = \bigvee \{ U\langle A, \alpha\rangle \rightarrowtail UHC \mid \langle A, \alpha\rangle \in \mathsf{Sub}_{\mathcal{C}_\Gamma}(HC) \}$$

$$\boxslash P = \bigvee \{ Q \rightarrowtail UHC \mid \forall h : HC \longrightarrow HC \,.\, \exists_h Q \leq P \}$$

The invariance theorem

$$\Box P = \bigvee\{U\langle A,\, \alpha\rangle \rightarrowtail UHC \mid \langle A,\, \alpha\rangle \in \mathsf{Sub}_{\mathcal{C}_\Gamma}(HC)\}$$

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- $\Box P$ is the (carrier of the) largest subcoalgebra of HC.

The invariance theorem

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- $\Box P$ is the (carrier of the) largest subcoalgebra of HC.
- $\boxslash P$ is the largest endomorphism invariant subobject of UHC, that is:
  - For every $h : HC \to HC$, $\exists_h \boxslash P \leq \boxslash P$;

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- $\boxslash P$ is the largest endomorphism invariant subobject of UHC, that is:
  - For every $h : HC \rightarrow HC$, $\exists_h \boxslash P \leq \boxslash P$;
  - If, for every $h : HC \rightarrow HC$, $\exists_h Q \leq Q$, then $Q \leq P$.

The invariance theorem

$$\Box P = \bigvee\{U\langle A,\,\alpha\rangle \rightarrowtail UHC \mid \langle A,\,\alpha\rangle \in \mathsf{Sub}_{\mathcal{C}_\Gamma}(HC)\}$$

$$\boxslash P = \bigvee\{Q \rightarrowtail UHC \mid \forall h\!:\!HC \longrightarrow HC\,.\,\exists_h Q \leq P\}$$

$\boxslash$ is an **S4** necessity operator.

- If $P \vdash Q$ then $\boxslash P \vdash \boxslash Q$;
- $\boxslash P \vdash P$;
- $\boxslash P \vdash \boxslash\,\boxslash\,P$;
- $\boxslash(P \to Q) \vdash \boxslash P \to \boxslash Q$;

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- $\boxslash P \vdash \boxslash\boxslash P$;
- $\boxslash(P \to Q) \vdash \boxslash P \to \boxslash Q$;

If Γ preserves pullbacks of $\mathcal{S}$-morphisms, then so is $\Box$.

The invariance theorem

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Theorem (The invariance theorem). *Let φ be a coequation over C. For any coequation ψ over C, $\mathsf{Mod}(\varphi) \models \psi$ iff $\Box \boxslash \varphi \leq \psi$.*

The invariance theorem

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Theorem (The invariance theorem). *Let φ be a coequation over C. For any coequation ψ over C, $\mathsf{Mod}(\varphi) \models \psi$ iff $\Box \boxslash \varphi \leq \psi$.*

In other words, $\Box \boxslash P$ is the least coequation satisfied by $\mathsf{Mod}(P)$. It can be regarded as a measure of the "coequational commitment" of P.

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A sound rule

An inference rule $\dfrac{\varphi_1 \quad \ldots \quad \varphi_n}{\psi}$ is sound just in case, whenever $\langle A, \alpha \rangle \models \varphi_1, \ldots, \langle A, \alpha \rangle \models \varphi_n$, then $\langle A, \alpha \rangle \models \psi$.

A sound rule

An inference rule $\dfrac{\varphi_1 \quad \ldots \quad \varphi_n}{\psi}$ is sound just in case, whenever $\langle A, \alpha\rangle \models \varphi_1, \ldots, \langle A, \alpha\rangle \models \varphi_n$, then $\langle A, \alpha\rangle \models \psi$.

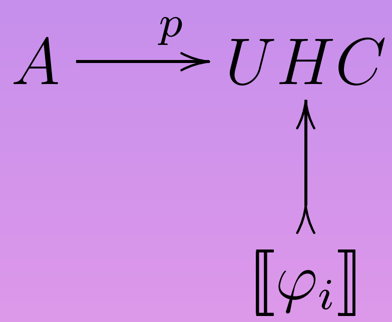
Theorem. *The rule* $\dfrac{\bigwedge \varphi_i}{\varphi_i} \bigwedge$*-E is sound.*

A sound rule

Theorem. $\bigwedge$*-E is sound.*

Proof. Suppose $\langle A, \alpha\rangle \models \bigwedge \varphi_i$ and $p\colon\langle A, \alpha\rangle \to HC$. We must show that $\mathsf{Im}(p) \leq [\![\varphi_i]\!]$.

$\ggg$



A sound rule

Theorem. *$\bigwedge$-E is sound.*

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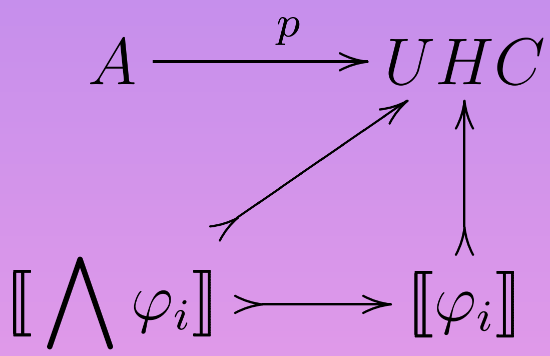
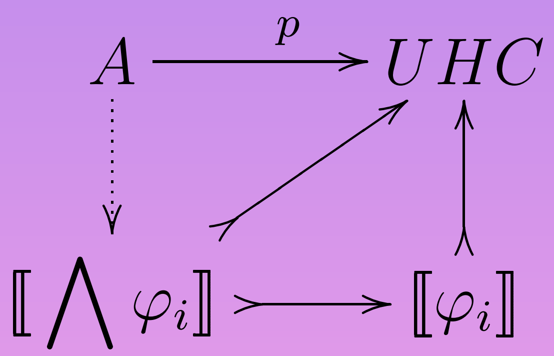


A sound rule

Theorem. $\bigwedge$*-E is sound.*

Proof. Suppose $\langle A, \alpha \rangle \models \bigwedge \varphi_i$ and $p : \langle A, \alpha \rangle \to HC$. We must show that $\mathsf{Im}(p) \leq [\![\varphi_i]\!]$. But we know $\mathsf{Im}(p) \leq [\![\bigwedge \varphi_i]\!] \leq [\![\varphi_i]\!]$. $\square$



A coequational calculus

The following rules are sound.

$$\frac{\bigwedge \varphi_i}{\varphi_i} \bigwedge\text{-E}$$

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The following rules are sound.

$$\frac{\bigwedge \varphi_i}{\varphi_i} \bigwedge\text{-E} \qquad\qquad \frac{\{\varphi_i\}_{i\in I}}{\bigwedge \varphi_i} \bigwedge\text{-I}$$

If $\mathsf{Im}(p\colon\langle A,\,\alpha\rangle\to HC) \leq [\![\varphi_i]\!]$ for each $i \in I$, then $\mathsf{Im}(p) \leq \bigwedge [\![\varphi_i]\!]$.

A coequational calculus

The following rules are sound.

$$\frac{\bigwedge \varphi_i}{\varphi_i} \bigwedge\text{-E} \qquad \frac{\{\varphi_i\}_{i\in I}}{\bigwedge \varphi_i} \bigwedge\text{-I}$$

$$\frac{\varphi}{\Box\varphi} \Box\text{-I}$$

If $\mathsf{Im}(p : \langle A,\, \alpha\rangle \rightarrow HC) \leq [\![\varphi]\!]$, then $\mathsf{Im}(p) \leq \Box[\![\varphi]\!]$ (because $\mathsf{Im}(p)$ is a subcoalgebra contained in φ).

A coequational calculus

The following rules are sound.

$$\frac{\bigwedge \varphi_i}{\varphi_i} \bigwedge\text{-E} \qquad \frac{\{\varphi_i\}_{i\in I}}{\bigwedge \varphi_i} \bigwedge\text{-I}$$

$$\frac{\varphi}{\Box\varphi} \Box\text{-I} \qquad \frac{\varphi}{\varphi(h(x))} \text{Sub}$$

Here, Sub applies for every Γ-homomorphism $h\colon HC \to HC$.

A coequational calculus

The following rules are sound.

$$\frac{\bigwedge \varphi_i}{\varphi_i} \bigwedge\text{-E} \qquad \frac{\{\varphi_i\}_{i \in I}}{\bigwedge \varphi_i} \bigwedge\text{-I}$$

$$\frac{\varphi}{\Box \varphi} \Box\text{-I} \qquad \frac{\varphi}{\varphi(h(x))} \text{Sub}$$

Let $p : HC \to HC$ be given.

$$\mathsf{Im}(p) \leq h^* [\![\varphi]\!] \text{ iff } \exists_h \, \mathsf{Im}(p) \leq [\![\varphi]\!].$$

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Let $p : HC \to HC$ be given.

$$\mathsf{Im}(p) \leq h^*[\![\varphi]\!] \text{ iff } \mathsf{Im}(h \circ p) \leq [\![\varphi]\!].$$

Hence, if for every $q : HC \to HC$, $\mathsf{Im}(q) \leq [\![\varphi]\!]$, then $\mathsf{Im}(p) \leq h^*[\![\varphi]\!]$.

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We call this rule DSR for Damn Semantic Rule. It is a damn shame that we've had to include such an ugly rule.

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We need this rule (along with $\bigwedge$-E) to ensure that the deductive closure of S is closed upwards, so if $[\![\varphi]\!] \leq [\![\psi]\!]$, then $\varphi \vdash \psi$.

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Maybe, we can replace this semantic rule with a rule $\dfrac{\varphi \qquad \varphi \vdash \psi}{\psi}$ where $\varphi \vdash \psi$ is proven in an appropriate logic for $\mathsf{Sub}(UHC)$.

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Let $S \subseteq \mathcal{L}_{\mathsf{Coeq}}$. Let $\mathsf{Ded}(S)$ denote the deductive closure of S under these rules. We see

$$\mathsf{Ded}(S) \subseteq \mathsf{Th}\,\mathsf{Mod}(S).$$

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A lemma

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$$\boxslash[\![\varphi]\!] = \bigwedge\{h^*[\![\varphi]\!] \mid h\colon HC \longrightarrow HC\}.$$

Proof. Recall $\boxslash[\![\varphi]\!] = \bigvee\{P \mid \forall h\colon HC \to HC \,.\, \exists_h P \leq [\![\varphi]\!]\}$.

$\supseteq$**:** It suffices to show that for all $k\colon HC \to HC$,

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$\subseteq$**:** It suffices to show that for all $k : HC \rightarrow HC$,

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But, $\boxslash [\![\varphi]\!]$ is invariant, so $\exists_k \boxslash [\![\varphi]\!] \leq \boxslash [\![\varphi]\!] \leq \varphi$.

A completeness theorem (of sorts)

Theorem. *Let* $S \subseteq \mathcal{L}_{\mathsf{Coeq}}$. *If* $\mathsf{Mod}(S) \models \varphi$, *then* $\varphi \in \mathsf{Ded}(S)$, *i.e.,* $\mathsf{Th}\,\mathsf{Mod}(S) \subseteq \mathsf{Ded}(S)$.

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Proof. So, we see that $S \vdash \Box \bigwedge \{\psi(h(x)) \mid h : HC \to HC\}$. Now, by the lemma,

$$[\![\Box \bigwedge \{\psi(h(x)) \mid h : HC \longrightarrow HC\}]\!] = \Box \boxslash [\![\psi]\!],$$

and by the Invariance Theorem, $\Box \boxslash [\![\psi]\!] \leq [\![\varphi]\!]$. ⋙

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Proof. Hence,

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Note: We used $\bigwedge$-E and DSR only to show that if $\Box \boxslash \psi \in S$, then $\varphi \in \mathsf{Ded}(S)$.

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Damn semantic rule!

Sketch of completeness

1. Define two operators $\mathsf{Sub}(UHC) \to \mathsf{Sub}(UHC)$:

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Note:

$$\begin{aligned}\mathsf{Mod}(S) &= \mathsf{Mod}(\{\varphi \Rightarrow \mathbf{cons}_S\,\varphi \mid \varphi \in \mathcal{L}_{\mathsf{Coeq}}\}) \\ &= \mathsf{Mod}(\{\varphi \Rightarrow \mathbf{ent}_S\,\varphi \mid \varphi \in \mathcal{L}_{\mathsf{Coeq}}\})\end{aligned}$$

Subgoal: Show $\mathbf{cons}_{\mathsf{Ded}(S)} = \mathbf{ent}_S$.

Sketch of completeness

1. Define two operators $\mathsf{Sub}(UHC) \rightarrow \mathsf{Sub}(UHC)$:
2. Show that $\mathbf{ent}_S$ is the greatest suboperator of $\Box \circ \mathbf{cons}_S$ such that:
   - $\mathbf{ent}_S$ is a comonad (deflationary, idempotent, monotone);
   - $\mathbf{ent}_S$ is *endomorphism invariant* – for all $h : HC \rightarrow HC$, $\exists_h \circ \mathbf{ent}_S \leq \mathbf{ent}_S \circ \exists_h$.

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2. Show that $\mathbf{ent}_S$ is the greatest **EIEIO**.
3. Show that if S is deductively closed, $\mathbf{cons}_S$ is **EIEIO**. Hence, $\mathbf{cons}_S = \mathbf{ent}_S$.
4. $\mathsf{Imp}\,\mathsf{Mod}(S) = \{\varphi \Rightarrow \psi \mid \psi \geq \mathbf{ent}_S\,\varphi\}$. Use DSR and $\bigwedge$-E to show that $\mathsf{Ded}(S) = \mathsf{Imp}\,\mathsf{Mod}(S)$.

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