Horn Covarieties for Coalgebras

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Outline

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II. Dual theorems for $\mathcal{E}_\Gamma$

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Equations in $\mathbf{Set}^\Gamma$

Let $\Gamma : \mathbf{Set} \to \mathbf{Set}$ be a polynomial functor and let $X \in \mathbf{Set}$ be regular projective (means nothing in **Set**!).

$$\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Set}^\Gamma \qquad (F \dashv U)$$

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We say $\langle A,\, \alpha\rangle \models t_1 =_X t_2$ iff for every $\sigma : X \to A$, we have $\widetilde{\sigma} \circ t_1 = \widetilde{\sigma} \circ t_2$.

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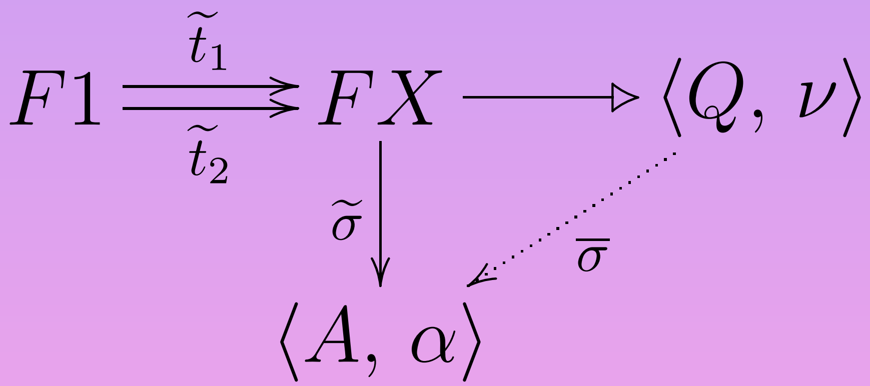
Let $\langle Q, \nu \rangle$ be the coequalizer of the congruence generated by $t_1 =_X t_2$.
$\langle A, \alpha \rangle \models t_1 =_X t_2$ iff for every $\widetilde{\sigma} : FX \to \langle A, \alpha \rangle$, there is a homomorphism $\overline{\sigma}$ making the diagram below commute.



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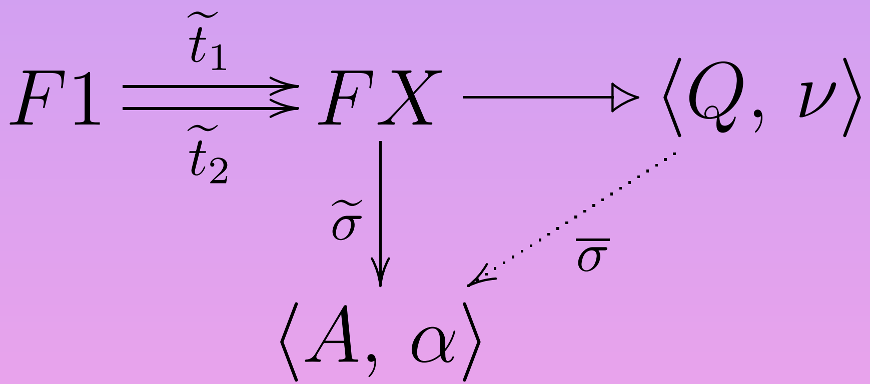

$$\mathsf{Hom}(X, A) \cong \mathsf{Hom}(FX, \langle A, \alpha\rangle) \cong \mathsf{Hom}(\langle Q, \nu\rangle, \langle A, \alpha\rangle)$$

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$$S \rightrightarrows UFX$$

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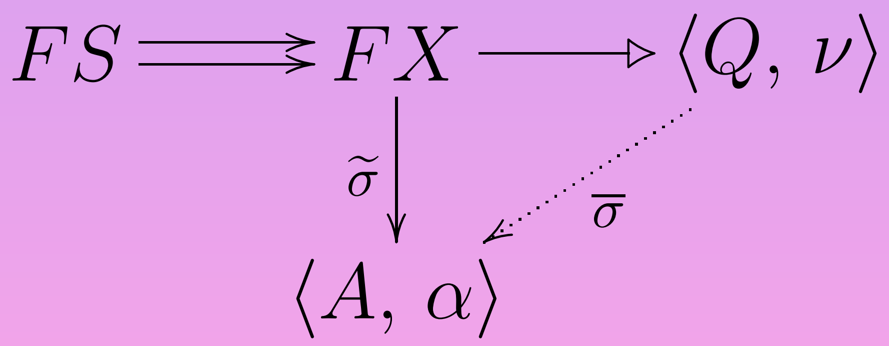

Conjunctions of equations

Let S be a set of equations over X, i.e., $S \subseteq UFX \times UFX$.
Let $\mathbf{V} \subseteq \mathbf{Set}^{\Gamma}$ and define

$$\mathsf{EqTh}(\mathbf{V}) = \{S \mid \exists \text{ reg. proj. } X \,.\, S \subseteq UFX \times UFX, \quad \mathbf{V} \models_X \bigwedge S\}.$$

Implications of equations

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Write $\langle A,\, \alpha\rangle \models_X \bigwedge S \Rightarrow \bigwedge T$ just in case, for every $\widetilde{\sigma} : FX \to \langle A,\, \alpha\rangle$, if $\widetilde{\sigma}$ coequalizes $S \rightrightarrows UFX$, then $\widetilde{\sigma}$ also coequalizes $T \rightrightarrows UFX$.

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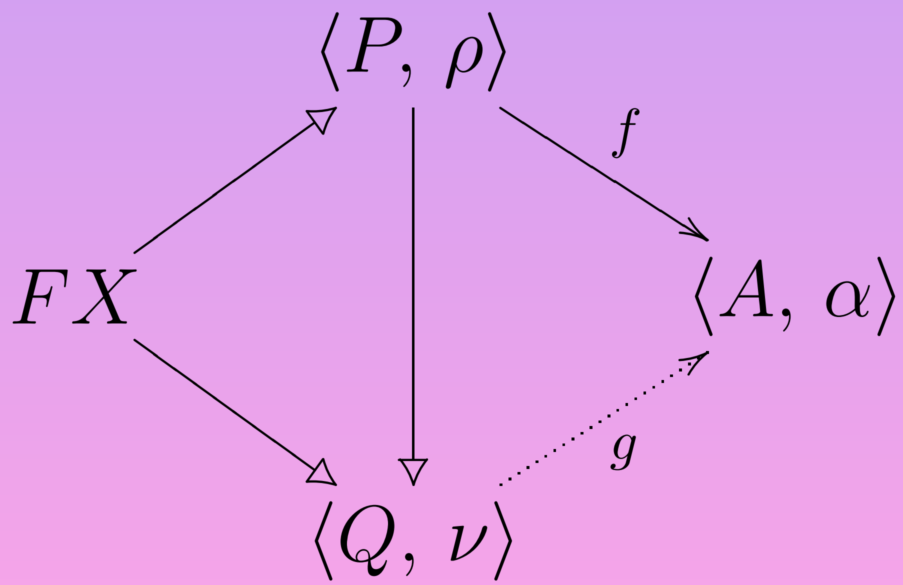
$\langle A,\, \alpha\rangle \models_X \bigwedge S \Rightarrow \bigwedge T$ iff for every $f\colon\langle P,\, \rho\rangle \to \langle A,\, \alpha\rangle$, there is a morphism $g\colon\langle Q,\, \nu\rangle \to \langle A,\, \alpha\rangle$ making the diagram below commute.



Implications of equations

Let S, T be sets of equations over X.
Let $\langle P, \rho\rangle$, $\langle Q, \nu\rangle$ be the coequalizer of the congruence generated by S, $S \cup T$, resp.
Equivalently, $\langle A, \alpha\rangle \models \bigwedge S \Rightarrow \bigwedge T$ just in case

$$\mathsf{Hom}(\langle P, \rho\rangle, \langle A, \alpha\rangle) \cong \mathsf{Hom}(\langle Q, \nu\rangle, \langle A, \alpha\rangle).$$

Implications of equations

Let S, T be sets of equations over X.
Define

$$\mathsf{ImpEqTh}(\mathbf{V}) = \{\langle S, T\rangle \mid \exists \text{ reg. proj. } X \,. \\ S, T \subseteq UFX \times UFX, \\ \mathbf{V} \models_X \bigwedge S \Rightarrow \bigwedge T\}.$$

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Note: $\mathsf{EqTh}(\mathbf{V}) \subseteq \mathsf{ImpEqTh}(\mathbf{V})$, via

$$S \ \mapsto \ \bigwedge \emptyset \Rightarrow \bigwedge S.$$

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Let $\langle P, \rho\rangle$ be the coequalizer of the congruence generated by S.
$\langle A, \alpha\rangle \models_X \neg \bigwedge S$ just in case there is no homomorphism $\langle P, \rho\rangle \to \langle A, \alpha\rangle$, i.e.,

$$\mathsf{Hom}(\langle P, \rho\rangle, \langle A, \alpha\rangle) = \emptyset.$$

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Let $\mathcal{S} \subseteq \mathsf{HornEqTh}$ $(= \mathsf{HornEqTh}(\emptyset))$. Define

$$\mathsf{Sat}(\mathcal{S}) = \{\langle A, \alpha\rangle \in \mathbf{Set}^{\Gamma} \mid \langle A, \alpha\rangle \models \mathcal{S}\}.$$

The H, S, P, P^+ operators

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The variety theorems

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Theorem (Birkhoff variety theorem).

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Theorem (Horn variety theorem).

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Closure operators for coalgebras

Recall the algebra operators.

$$H\mathbf{V} = \{\langle B, \beta\rangle \in \mathcal{E}^{\Gamma} \mid \exists \mathbf{V} \ni \langle C, \gamma\rangle \longrightarrow\!\!\!\!\triangleright \langle B, \beta\rangle\}$$

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Closure operators for coalgebras

Each algebra operator yields a coalgebra operator.

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$$\Sigma\mathbf{V} = \{\langle B, \beta\rangle \in \mathcal{E}_\Gamma \mid \exists \{\langle A_i, \alpha_i\rangle\}_{i\in I} \subseteq \mathbf{V}\,.$$

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Dualizing equations

Consider again equations in $\mathbf{Set}^{\Gamma}$. We consider the mapping

$$\{S \rightarrowtail UFX \times UFX\} \to \{FX \twoheadrightarrow \langle Q, \nu\rangle\},$$

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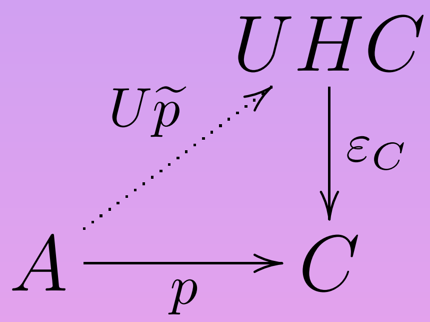
Return to $\mathcal{E}_{\Gamma}$. Let $\mathcal{E}$, Γ be good (co-good?) and let H be the right adjoint to $U\colon\mathcal{E}_{\Gamma}\to\mathcal{E}$, with counit $\varepsilon\colon UH\to 1$.

Dualizing equations

Return to $\mathcal{E}_\Gamma$. Let $\mathcal{E}$, Γ be good and let H be the right adjoint to $U:\mathcal{E}_\Gamma \to \mathcal{E}$, with counit $\varepsilon : UH \to 1$.

Reminder: Let $C \in \mathcal{E}$ and $\langle A, \alpha \rangle \in \mathcal{E}_\Gamma$. For any C-coloring $p : A \to C$ of A, there exists a unique homomorphism $\widetilde{p} : \langle A, \alpha \rangle \to HC$ making the diagram below commute.



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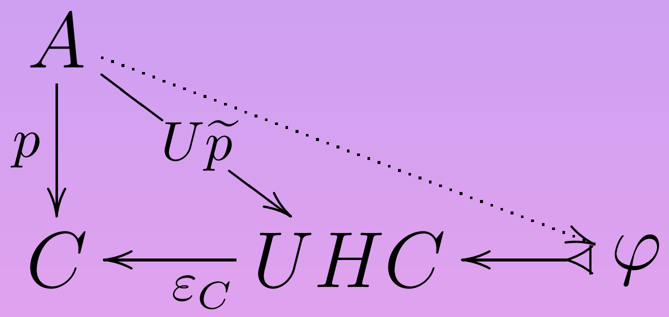
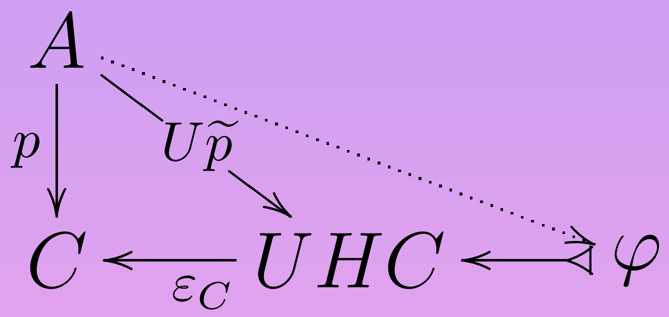
A *coequation* over C is a regular subobject $\varphi \leq UHC$.
We write $\langle A, \alpha \rangle \models_C \varphi$ iff for every coloring $p : A \to C$ of A, the adjoint transpose $U\widetilde{p}$ factors through φ.



Dualizing equations

Return to $\mathcal{E}_\Gamma$. Let $\mathcal{E}$, Γ be good and let H be the right adjoint to $U : \mathcal{E}_\Gamma \twoheadrightarrow \mathcal{E}$, with counit $\varepsilon : UH \twoheadrightarrow 1$.
A *coequation* over C is a regular subobject $\varphi \leq UHC$.
We write $\langle A, \alpha \rangle \models_C \varphi$ iff for every coloring $p : A \twoheadrightarrow C$ of A, $\mathsf{Im}(U\widetilde{p}) \leq \varphi$.



In other words,

$$\mathsf{Hom}(A, C) \cong \mathsf{Hom}(\langle A, \alpha \rangle, HC) \cong \mathsf{Hom}(\langle A, \alpha \rangle, \Box\varphi).$$

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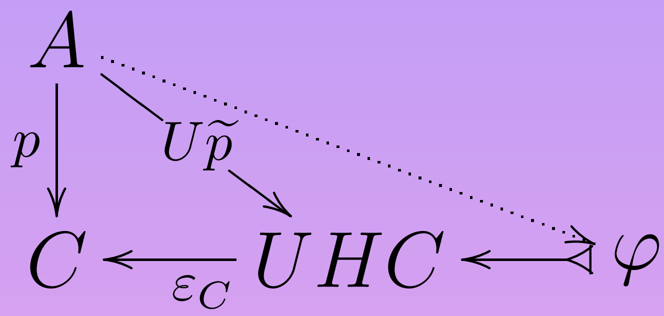

$\langle A,\, \alpha \rangle \models_C \varphi$ just in case, however we paint the elements of A, they “look like” elements of φ.

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We view coequations φ as predicates over UHC.

$\langle A, \alpha \rangle \models_C \varphi$ iff, for every $p : A \twoheadrightarrow C$, we have

$$\mathsf{Im}(U\widetilde{p}) \vdash \varphi.$$

Conditional coequations

Let $\varphi,\ \psi \leq UHC$.
We write $\langle A,\ \alpha\rangle \models \varphi \Rightarrow \psi$ just in case, for every $p\colon A \to C$ such that $\mathsf{Im}(\widetilde{p}) \leq \varphi$, we have $\mathsf{Im}(\widetilde{p}) \leq \psi$.

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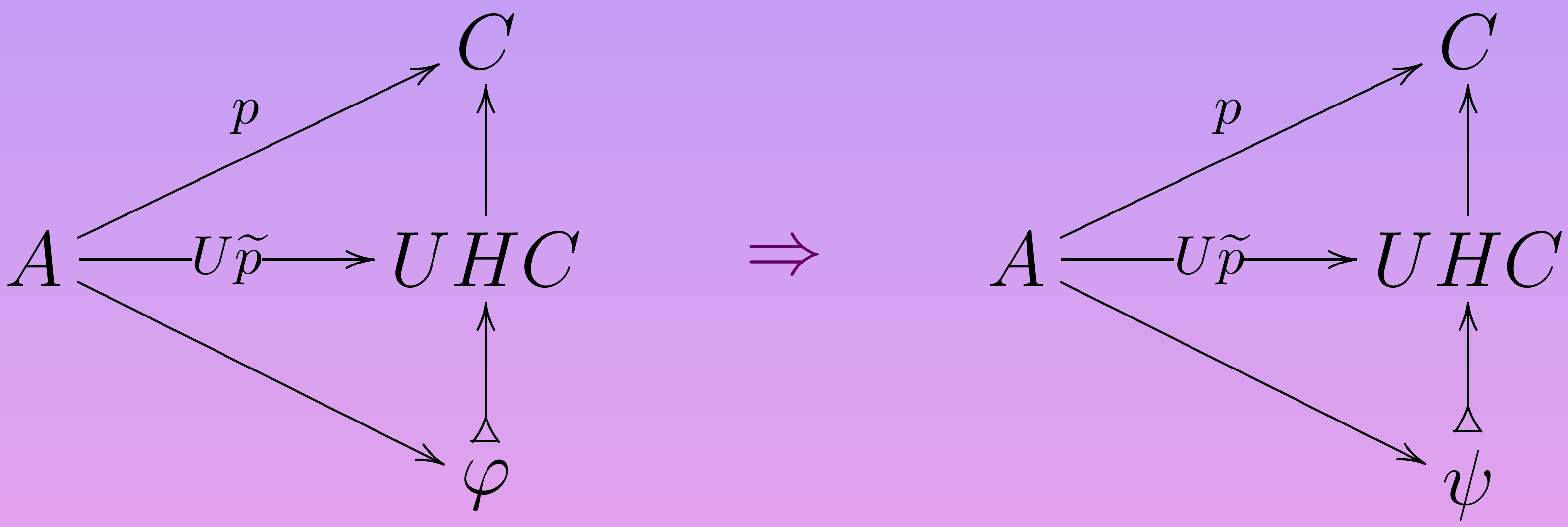

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$\langle A, \alpha\rangle \models \varphi \Rightarrow \psi$ just in case every homomorphism $\langle A, \alpha\rangle \to \Box\varphi$ factors through $\Box\psi$, i.e.,

$$\mathsf{Hom}(\langle A, \alpha\rangle, \Box\varphi) \cong \mathsf{Hom}(\langle A, \alpha\rangle, \Box\psi).$$

Dualizing negations

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No matter how we paint A, there is some element $a \in A$ that doesn't land in φ.

Note: This does not mean that $\langle A, \alpha \rangle \models \neg\varphi$! "Something in A does not land in φ," is not the same as, "Everything in A does not land in φ."

A few more things...

Let $\mathbf{V} \subseteq \mathcal{E}_\Gamma$.

$$\mathsf{CoeqTh}(\mathbf{V}) = \{\varphi \mid \exists \text{ reg. inj. } C \,.\, \varphi \leq UHC, \\ \mathbf{V} \models_C \varphi\}$$

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\mathsf{HornCoeqTh}(\mathbf{V}) &= \mathsf{ImpCoeqTh}(\mathbf{V}) \cup \\
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$$\mathsf{HornCoeqTh}(\mathbf{V}) = \mathsf{ImpCoeqTh}(\mathbf{V}) \cup \{\overline{\varphi} \mid \exists \text{ reg. inj. } C \,.\, \varphi \leq UHC, \; \mathbf{V} \models_C \overline{\varphi}\}$$

Let $\mathcal{S} \subseteq \mathsf{HornCoeqTh}$. Define

$$\mathsf{Sat}(\mathcal{S}) = \{\langle A, \alpha\rangle \in \mathcal{E}_\Gamma \mid \langle A, \alpha\rangle \models \mathcal{S}\}.$$

The covariety theorems

Let $\mathcal{E}$, Γ be good and $\mathbf{V} \subseteq \mathcal{E}$.

Theorem (Birkhoff covariety theorem).

$$\mathsf{Sat}(\mathsf{CoeqTh}\,\mathbf{V}) = SH\Sigma\mathbf{V}$$

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Theorem (Horn covariety theorem).

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Some simple examples

Fix a set Z and consider $\Gamma : \mathbf{Set} \to \mathbf{Set}$ where $\Gamma X = Z \times X$.
Regard a Γ-coalgebra $\langle A, \alpha \rangle$ as a set of streams over Z and let

$$h_\alpha : A \longrightarrow Z$$

$$t_\alpha : A \longrightarrow A$$

denote the evident head and tail operations.

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- $\{\langle A, \alpha \rangle \in \mathbf{Set}_\Gamma \mid A \neq \emptyset$ and $\forall a \in A \, \exists n \in \mathbb{N} \,.\, t_\alpha^n(a) = t_\alpha^{n+1}(a)\}$.

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- $\{\langle A, \alpha \rangle \in \mathbf{Set}_\Gamma \mid A \neq \emptyset$ and $\forall a \in A \,\exists n \in \mathbb{N} \,\forall m > n \,.\, h_\alpha \circ t_\alpha^n(a) = h_\alpha \circ t_\alpha^m(a)\}$.

Deterministic automata and languages

Fix an alphabet $\mathcal{I}$. Let

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A Γ-coalgebra $\langle A, \alpha \rangle$ is an automaton accepting input from $\mathcal{I}$ and outputting either 0 or 1, where

$$\mathsf{out}_{\alpha}(a) = \pi_1 \circ \alpha(a)$$

$$\mathsf{trans}_{\alpha}(a) = \pi_2 \circ \alpha(a)$$

Deterministic automata and languages

Let $\sigma \in \mathcal{I}^{<\omega}$ and define

$$\mathsf{eval}_\alpha : A \times \mathcal{I}^{<\omega} \longrightarrow A$$

by

$$\mathsf{eval}_\alpha(a, ()) = a,$$
$$\mathsf{eval}_\alpha(a, \sigma * i) = \mathsf{trans}_\alpha(\mathsf{eval}_\alpha(a, \sigma))(i).$$

$\mathsf{eval}_\alpha(a, \sigma)$ is the final state of the calculation beginning in a with input σ.

Deterministic automata and languages

Define

$$\mathsf{acc}_\alpha : A \longrightarrow \mathcal{P}(\mathcal{I}^{<\omega})$$

by

$$\mathsf{acc}_\alpha(a) = \{\sigma \in \mathcal{I}^{<\omega} \mid \mathsf{out}_\alpha \circ \mathsf{eval}_\alpha(a, \sigma) = 1\}.$$

$\mathsf{acc}_\alpha(a)$ is the set of all words accepted by state a.

Deterministic automata and languages

Fix a “language” $\mathcal{L} \subseteq \mathcal{I}^{<\omega}$ and define

$$\mathbf{V}_{\mathcal{L}} = \{\langle A, \alpha\rangle \in \mathbf{Set}_{\Gamma} \mid \exists a \in A\,.\ \mathsf{acc}_{\alpha}(a) = \mathcal{L}\}.$$

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Explicitly: the class of all automata which have an initial state accepting exactly $\mathcal{L}$ is closed under codomains of epis and non-empty coproducts. Furthermore, $\mathbf{V}_{\mathcal{L}}$ is definable by a Horn coequation.

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Indeed, let $\varphi \leq UH1$ be the set

$$\{c \in UH1 \mid \mathsf{acc}_{H1}(c) \neq \mathcal{L}\}.$$

Then $\langle A, \alpha\rangle \in \mathbf{V}_{\mathcal{L}}$ just in case

$$\mathsf{Hom}(\langle A, \alpha\rangle, \Box\varphi) = \emptyset.$$

More automata

Fix $\mathcal{L} \subseteq \mathcal{I}^{<\omega}$.

1. Deterministic automata which have an accepting state for $\mathcal{L}$.

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Fix $\mathcal{L} \subseteq \mathcal{I}^{<\omega}$.

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4. Etc. and so on.

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Cofree for $H\Sigma^{+}\mathbf{V}$ coalgebras

Let $\mathbf{V} \subseteq \mathcal{E}_{\Gamma}$, $C \in \mathcal{E}$. Define

$$\Theta C = \{f : \langle A, \alpha \rangle \longrightarrow HC \mid \langle A, \alpha \rangle \in \mathbf{V}\},$$

$$\Delta C = \bigvee \{\mathsf{Im}\, f \mid f \in \Theta C\}.$$

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Cofree for $H\Sigma^{+}\mathbf{V}$ coalgebras

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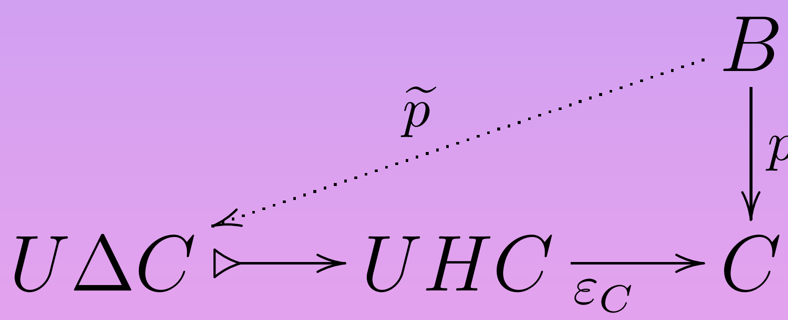
- $\Delta C \in H\Sigma^{+}\mathbf{V}$;
- If $\langle B, \beta\rangle \in H\Sigma^{+}\mathbf{V}$, then for every $p : B \to C$, there is a unique homomorphism $\widetilde{p} : \langle B, \beta\rangle \to \Delta C$ such that the diagram below commutes.



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If $\mathcal{E} = \mathbf{Set}$ (or any category in which each $C \neq 0$ has a global element) and $\mathbf{V} \neq 0$, then every $C \neq 0$ has a cofree for $H\Sigma^+\mathbf{V}$ coalgebra.

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In technical terms, we have *damn near an adjunction*. Indeed, it arises as the composition of an adjunction and *damn near a regular mono-coreflection*.

$$\mathcal{E} \underset{U}{\overset{H}{\rightleftarrows}} \mathcal{E}_\Gamma \rightleftarrows \mathbf{V}$$

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Behavioral classes

Consider the following operators.

$$R\mathbf{V} = \{\langle B,\, \beta\rangle \in \mathcal{E}_\Gamma \mid \exists \langle B,\, \beta\rangle \longrightarrow\!\!\!\!\!\!\rightarrow \langle A,\, \alpha\rangle \in \mathbf{V}\}$$

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Behavioral classes

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If, in $\mathcal{E}$, epis are stable under pullback, then also

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$$\mathsf{Sat}(\mathsf{CoeqTh}_{H1}\,\mathbf{V}) = RSH\Sigma\mathbf{V}$$

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Here, $\mathsf{CoeqTh}_{H1}\,\mathbf{V}$ ($\mathsf{ImpCoeqTh}_{H1}\,\mathbf{V}$, $\mathsf{HornCoeqTh}_{H1}\,\mathbf{V}$, resp.) denotes the (conditional, Horn, resp.) coequations over 1 color satisfied by $\mathbf{V}$.

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A slew of questions

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- Generic other stuff that won’t fit in the margin

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