Some Co-Birkhoff-Type Theorems

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Outline

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II. Equations and injectivity

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III. Injectivity and cones

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Birkhoff-type theorems

Let Γ be polynomial and $\mathbf{V} \subseteq \mathbf{Set}^{\Gamma}$.

Theorem (Birkhoff variety theorem).

$$\mathsf{Mod}\,\mathsf{Th}\,\mathbf{V} = \mathcal{HSP}\mathbf{V}$$

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Theorem (Horn variety theorem).

$$\mathsf{Mod}\,\mathsf{Horn}\,\mathbf{V} = \mathcal{SP}^{+}\mathbf{V}$$

Equations in $\mathbf{Set}^\Gamma$

Let $\Gamma : \mathbf{Set} \to \mathbf{Set}$ be a polynomial functor and let $X \in \mathbf{Set}$. We have an adjunction

$$\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Set}^\Gamma \quad (F \dashv U)$$

An *equation* over X is a pair $t_1 =_X t_2$ of elements of UFX, the carrier of the free algebra over X.

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We say $\langle A, \alpha\rangle \models t_1 =_X t_2$ iff for every $\sigma : X \to A$, we have $\widetilde{\sigma} \circ t_1 = \widetilde{\sigma} \circ t_2$.

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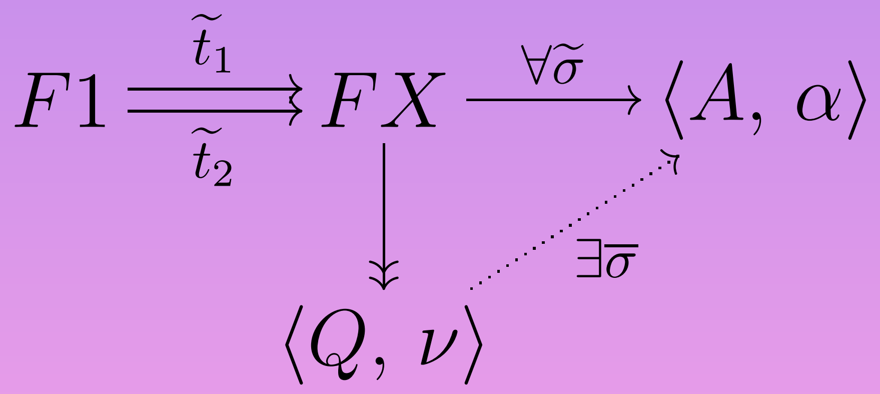
Let $\langle Q, \nu \rangle$ be the coequalizer of $F1 \underset{\widetilde{t_2}}{\overset{\widetilde{t_1}}{\rightrightarrows}} FX$.

$\langle A, \alpha \rangle \models t_1 =_X t_2$ iff for every $\widetilde{\sigma} : FX \to \langle A, \alpha \rangle$, there is a homomorphism $\overline{\sigma}$ making the diagram below commute.



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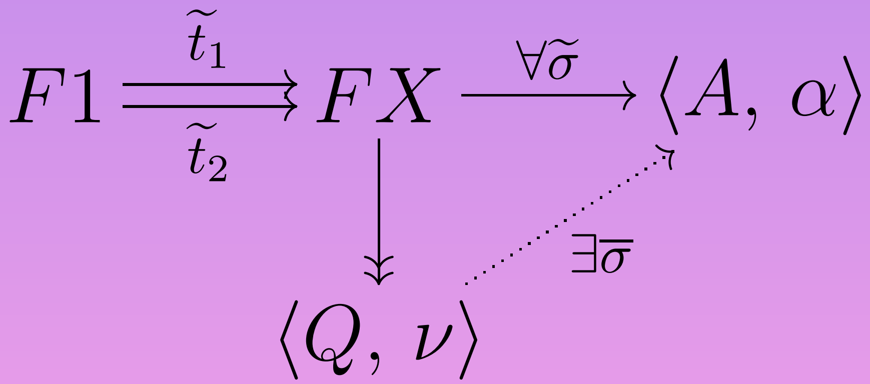

$$\mathsf{Hom}(X, A) \cong \mathsf{Hom}(FX, \langle A,\, \alpha \rangle) \cong \mathsf{Hom}(\langle Q,\, \nu \rangle, \langle A,\, \alpha \rangle)$$

Sets of equations

Consider a set E of equations over X and say $\langle A, \alpha\rangle \models E$ iff $\langle A, \alpha\rangle \models t_1 = t_2$ for every $t_1 = t_2 \in E$.

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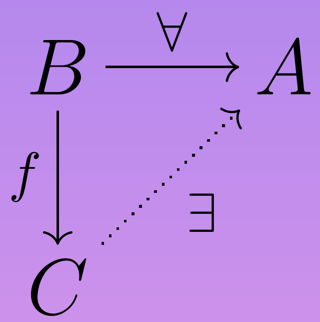
Let $q : FX \twoheadrightarrow \langle Q,\, \nu \rangle$ be the coequalizer.
Then $\langle A,\, \alpha \rangle \models E$ just in case every $FX \rightarrow \langle A,\, \alpha \rangle$ factors through q.

$$\begin{array}{ccccc} FE & \rightrightarrows & FX & \xrightarrow{\ \forall\ } & \langle A,\, \alpha \rangle \\ & & \downarrow\!\!\downarrow & \nearrow_{\exists} & \\ & & \langle Q,\, \nu \rangle & & \end{array}$$

Injectivity

Let $f : B \to C$ be given and $A \in \mathcal{C}$. We say that A is *f-injective* if, for every map $B \to A$ factors through f (not necessarily uniquely).



Injectivity

$$\begin{array}{ccc} B & \xrightarrow{\forall} & A \\ {\scriptstyle f}\big\downarrow & \overset{\exists}{\nearrow} & \\ C & & \end{array}$$

Let $FE \rightrightarrows FX \twoheadrightarrow \langle Q,\, \nu \rangle$ be a coequalizer diagram. $\langle A,\, \alpha \rangle \models E$ iff every $FX \to \langle A,\, \alpha \rangle$ factors through $\langle Q,\, \nu \rangle$.

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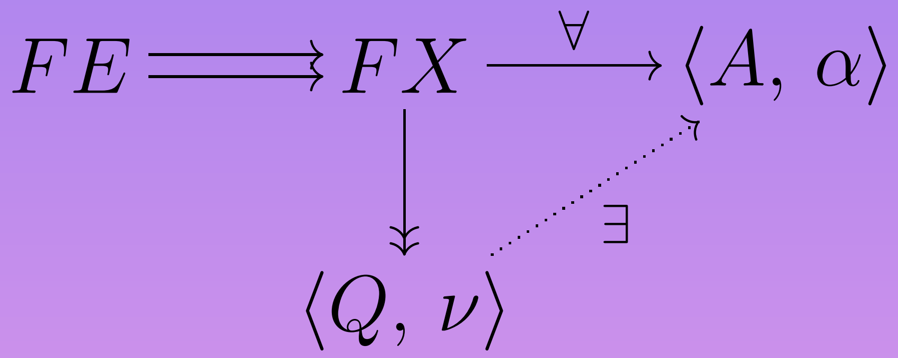
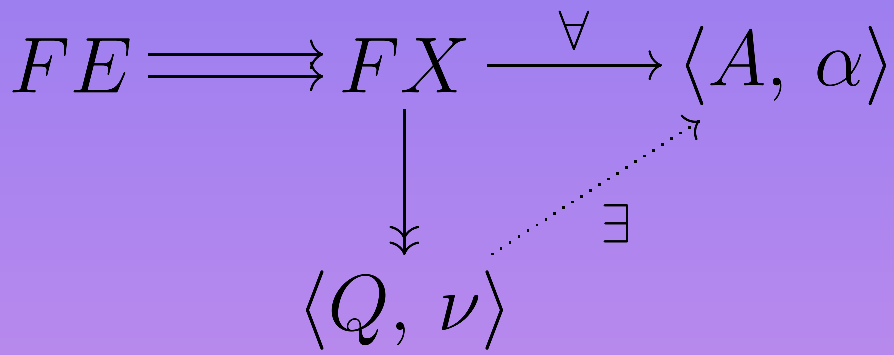


$\langle A,\, \alpha \rangle \models E$ just in case $\langle A,\, \alpha \rangle$ is injective with respect to $FX \rightarrow \langle Q,\, \nu \rangle$.

Injectivity



$\langle A,\, \alpha \rangle \models E$ just in case $\langle A,\, \alpha \rangle$ is injective with respect to $FX \to \langle Q,\, \nu \rangle$.

Thus, injectivity with respect to certain (classes of) arrows gives a notion of generalized equational satisfaction.

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Cone injectivity

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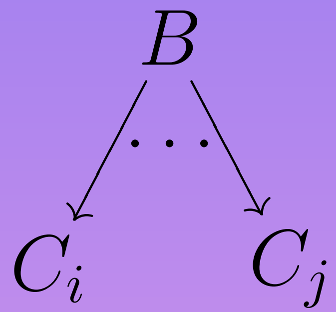

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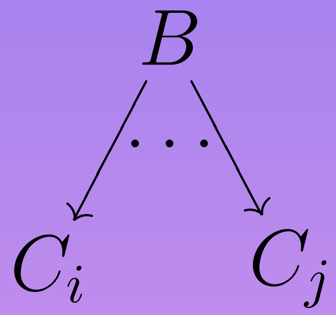
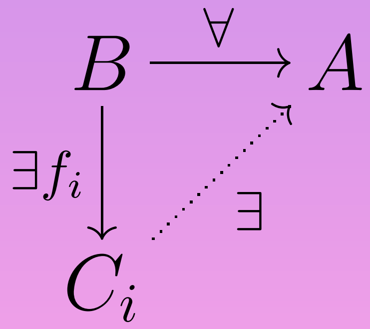


An object A is *injective* with respect to c if every $B \to A$ factors through some f_i.



The game plan

Németi and Sain defined, for each composition $\vec{\mathcal{X}} = \mathcal{HS}\Sigma$, $\mathcal{HS}\Sigma^{+}$, etc., a class of cones, $M_{\vec{\mathcal{X}}} \subseteq \mathsf{SubCat}(\mathsf{Cone}(\mathcal{C}))$.

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Each cone is a single arrow.

$$\begin{array}{c} B \\ \downarrow \\ C \end{array}$$

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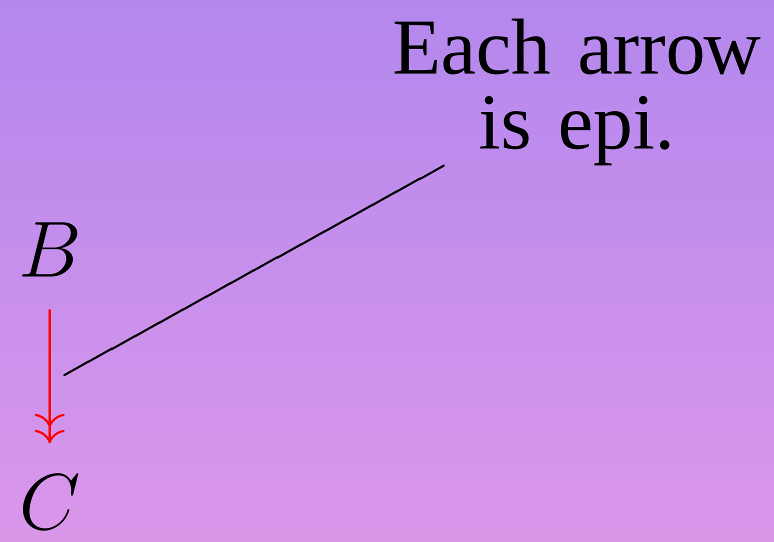

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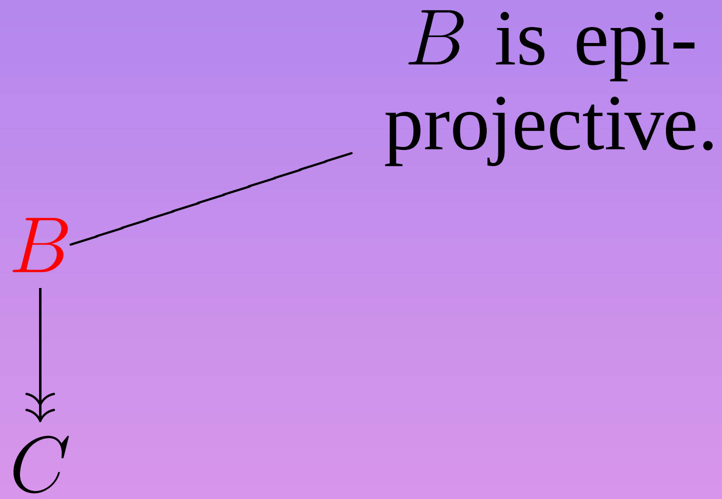

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Next, we define, for each $\vec{\mathcal{X}}$, an operator

$$K_{\vec{\mathcal{X}}} : \mathsf{SubCat}(\mathcal{C}) \longrightarrow \mathsf{SubCat}(\mathsf{Cocone}(\mathcal{C})).$$

$K_{\vec{\mathcal{X}}}\mathbf{V}$ represents the $M_{\vec{\mathcal{X}}}$-theory of $\mathbf{V}$. That is,

$$K_{\vec{\mathcal{X}}}\mathbf{V} = \{c \in M_{\vec{\mathcal{X}}} \mid \mathbf{V} \subseteq \mathbf{Inj}(c)\}.$$

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Finally, we prove a whole slew of theorems of the form

$$\mathbf{Inj}(M_{\vec{\mathcal{X}}}\mathbf{V}) = \vec{\mathcal{X}}\mathbf{V},$$

greatly impressing everybody.

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Plan B: Turn all the arrows around and see what you get. Hope someone is mildly interested.

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The abstract setting

We assume the following:

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- $\mathcal{C}$ has a factorization system $\langle \mathcal{H}, \mathcal{S} \rangle$.

This assumption appeared earlier in our use of epis. Implicitly, we were using the factorization system $\langle \mathrm{Epi}, \mathrm{Mono} \rangle$ in $\mathbf{Set}$.

The abstract setting

We assume the following:

- $\mathcal{C}$ has all coproducts.
- $\mathcal{C}$ has a factorization system $\langle \mathcal{H}, \mathcal{S} \rangle$.
- $\mathcal{C}$ is $\mathcal{S}$-well-powered

A category is *$\mathcal{S}$-well-powered* if for each $C \in \mathcal{C}$, the collection

$$\{j \in \mathcal{S} \mid \mathsf{cod}(j) = C\}/\cong$$

is a set.

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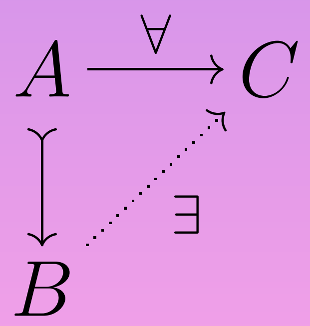
- $\mathcal{C}$ is $\mathcal{S}$-well-powered
- $\mathcal{C}$ has enough $\mathcal{S}$-injectives.

Recall an object C is $\mathcal{S}$-injective if, for all $A \rightarrowtail B$ in $\mathcal{C}$, and all $A \to C$, there is an extension $B \to C$.



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In **Set**, every non-empty set is Mono-injective.

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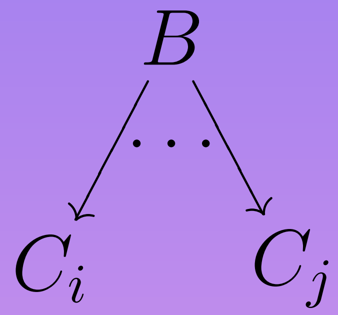
$\mathcal{C}$ has *enough injectives* if for every A in $\mathcal{C}$, there is an $\mathcal{S}$-injective C and a $\mathcal{S}$-morphism $A \rightarrowtail C$.

Projectivity and cocones

A *discrete cone* is a pair $c = \langle B, \{f_i : B \to C_i\}_{i \in I} \rangle$.



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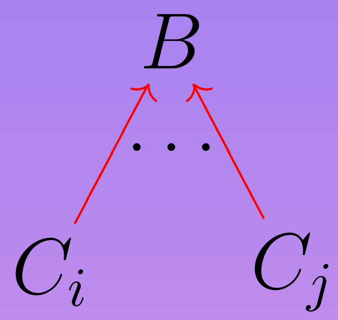

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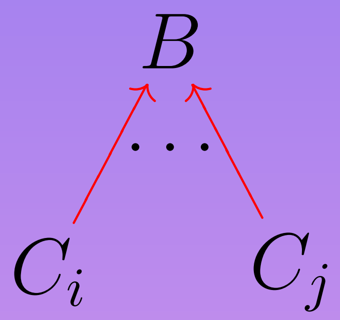

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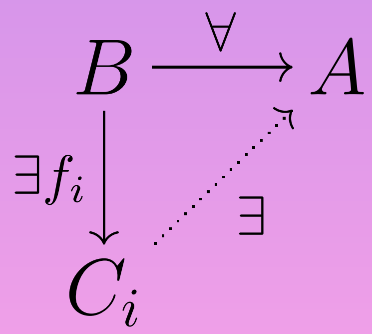

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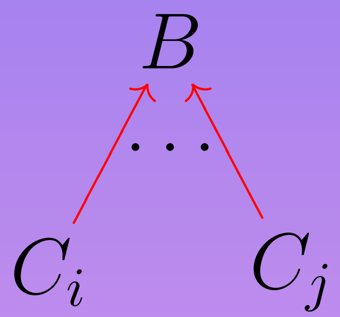
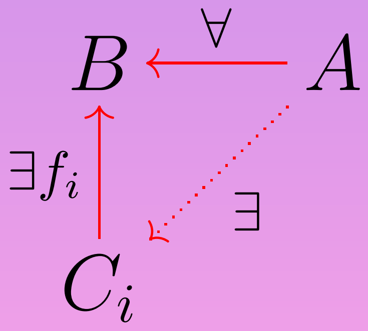
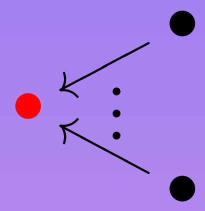


An object A is *projective* with respect to c if every $A \to B$ (co-)factors through some f_i.



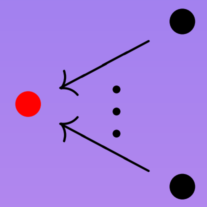
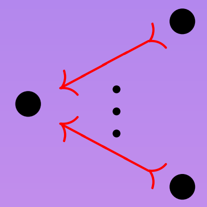
The cocone classes $M_{\vec{\chi}}$

Define

$M_{\mathcal{S}}$ cocones with injective vertex 

The cocone classes $M_{\vec{\mathcal{X}}}$

Define

$M_{\mathcal{S}}$	cocones with injective vertex	
$M_{\mathcal{H}}$	cocones with $\mathcal{S}$-morphisms	

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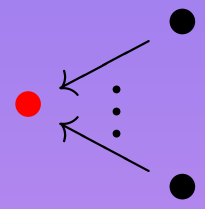
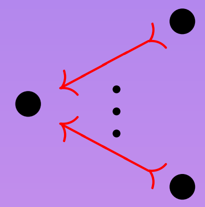
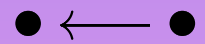
$M_{\mathcal{S}}$ cocones with injective vertex 

$M_{\mathcal{H}}$ cocones with $\mathcal{S}$-morphisms 

M_{Σ} cocones with one arrow 

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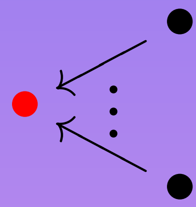

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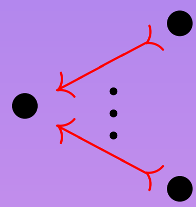
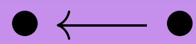
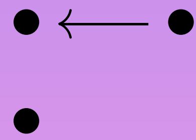


M_{Σ} cocones with one arrow



$M_{\Sigma+}$ cocones with 0 or 1 arrow



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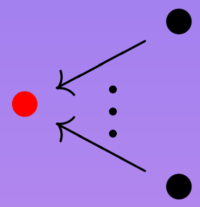
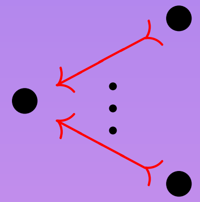
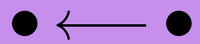
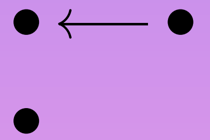
Define

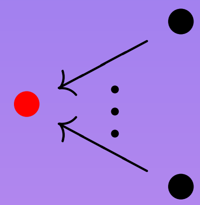
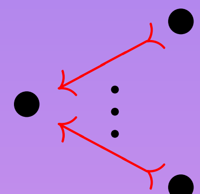
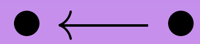
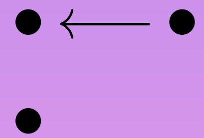
$M_{\mathcal{S}}$	cocones with injective vertex	
$M_{\mathcal{H}}$	cocones with $\mathcal{S}$-morphisms	
M_{Σ}	cocones with one arrow	
M_{Σ^+}	cocones with 0 or 1 arrow	

For composites $\vec{\mathcal{X}} = \mathcal{X}_1 \ldots \mathcal{X}_n$,

$$M_{\vec{\mathcal{X}}} = M_{\mathcal{X}_1} \cap \ldots \cap M_{\mathcal{X}_n}.$$

The cocone classes $M_{\vec{\mathcal{X}}}$

Define

$M_{\mathcal{S}}$	cocones with injective vertex	
$M_{\mathcal{H}}$	cocones with $\mathcal{S}$-morphisms	
M_{Σ}	cocones with one arrow	
$M_{\Sigma+}$	cocones with 0 or 1 arrow	

$M_{\vec{\mathcal{X}}}$ can be considered the language of the theory at hand.

Outline

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I. Some Birkhoff-type theorems
II. Equations and injectivity
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$$\mathsf{SubCat}(\mathcal{C}) \longrightarrow \mathsf{SubCat}(\mathcal{C}).$$

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Note: The symbols $\mathcal{H}$ and $\mathcal{S}$ do double duty, as classes of arrows and also as closure operators.

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A slew of theorems

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$$\mathbf{Proj}(K_{\vec{\mathcal{X}}}\mathbf{V}) = \vec{\mathcal{X}}\mathbf{V}$$

Here, $K_{\vec{\mathcal{X}}}\mathbf{V} = \{c \in M_{\mathbf{V}} \mid \mathbf{V} \subseteq \mathbf{Proj}(c)\}$

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Compare: $\mathsf{Mod}\,\mathsf{Th}\,\mathbf{V} = \mathcal{H}\mathcal{S}\mathcal{P}\mathbf{V}$ (Birkhoff)

Outline

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Categories of coalgebras

Let $\mathcal{C}$ satisfy our previous requirements and $\Gamma:\mathcal{C}\to\mathcal{C}$ be given. Let $U:\mathcal{C}_\Gamma\to\mathcal{C}$ be the forgetful functor.

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Thus, if Γ preserves $\mathcal{S}$-morphisms and $\mathcal{C}_\Gamma$ has cofree coalgebras, then $\mathcal{C}_\Gamma$ satisfies our abstract setting.

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Moreover, we may restrict our attention to cocones with cofree vertices, in the case that $\vec{\mathcal{X}}$ contains $\mathcal{S}$.

Deterministic automata and languages

Fix an alphabet $\mathcal{I}$. Let

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be the functor

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A Γ-coalgebra $\langle A, \alpha \rangle$ is an automaton accepting input from $\mathcal{I}$ and outputting either 0 or 1, where

$$\mathsf{out}_\alpha(a) = \pi_1 \circ \alpha(a)$$

$$\mathsf{trans}_\alpha(a) = \pi_2 \circ \alpha(a)$$

Deterministic automata and languages

Let $\sigma \in \mathcal{I}^{<\omega}$ and define

$$\mathsf{eval}_\alpha : A \times \mathcal{I}^{<\omega} \longrightarrow A$$

by

$$\begin{aligned} \mathsf{eval}_\alpha(a, ()) &= a, \\ \mathsf{eval}_\alpha(a, \sigma * i) &= \mathsf{trans}_\alpha(\mathsf{eval}_\alpha(a, \sigma))(i). \end{aligned}$$

$\mathsf{eval}_\alpha(a, \sigma)$ is the final state of the calculation beginning in a with input σ.

Deterministic automata and languages

Define

$$\mathsf{acc}_\alpha : A \longrightarrow \mathcal{P}(\mathcal{I}^{<\omega})$$

by

$$\mathsf{acc}_\alpha(a) = \{\sigma \in \mathcal{I}^{<\omega} \mid \mathsf{out}_\alpha \circ \mathsf{eval}_\alpha(a, \sigma) = 1\}.$$

$\mathsf{acc}_\alpha(a)$ is the set of all words accepted by state a.

Some classes of automata

Fix a language $\mathcal{L} \subseteq \mathcal{I}^{<\omega}$.

$\mathbf{V}\{\langle A, \alpha\rangle \mid \ldots\}$	$\mathbf{V}$ closed under
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$\exists! a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$	$\mathcal{H}$
$\exists! a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$ and $\forall b \in A\,.\ b \stackrel{*}{\longrightarrow} a$	$\mathcal{S}\mathcal{H}$

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$\exists! a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$ and $\forall b \in A\,.\ b \xrightarrow{*} a$	$\mathcal{SH}$

In fact, there's a "hidden" closure operator here.

Some classes of automata

Fix a language $\mathcal{L} \subseteq \mathcal{I}^{<\omega}$.

$\mathbf{V}\{\langle A, \alpha\rangle \mid \dots\}$	**V** closed under
$\forall a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$	$\mathcal{H}^{-}\mathcal{S}\mathcal{H}\Sigma$
$A \neq \emptyset \Rightarrow \exists a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$	$\mathcal{H}^{-}\mathcal{H}\Sigma$
$\exists a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$	$\mathcal{H}^{-}\mathcal{H}\Sigma^{+}$
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The $\mathcal{H}^{-}$ operator closes a class of coalgebras under domains of $\mathcal{H}$-morphisms.

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Behavioral classes

Consider the following operators.

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$$B\mathbf{V} = \{B \in \mathcal{C} \mid \exists \text{ relation } B \twoheadleftarrow R \twoheadrightarrow A \in \mathbf{V}\}$$

Here, a relation is an $\mathcal{S}$-morphism $R \rightarrowtail B \times A$ (we assume that $\mathcal{C}$ has finite products).

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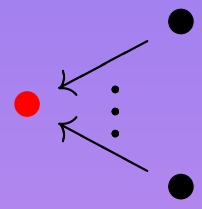
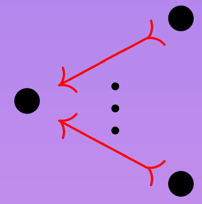
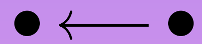
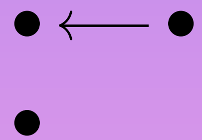
If, in $\mathcal{E}$, epis are stable under pullback, then also

$$\mathcal{H}^{-}\mathcal{H}\mathbf{V} = B\mathbf{V} = Q\mathbf{V}.$$

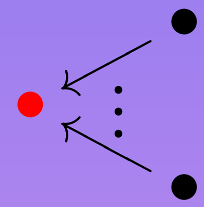
The cocone classes $M_{\vec{\mathcal{X}}}$

Recall

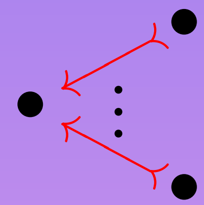
$M_{\mathcal{S}}$	cocones with injective vertex	
$M_{\mathcal{H}}$	cocones with $\mathcal{S}$-morphisms	
M_{Σ}	cocones with one arrow	
M_{Σ^+}	cocones with 0 or 1 arrow	

The cocone classes $M_{\vec{\mathcal{X}}}$

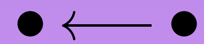
$M_{\mathcal{S}}$ cocones with injective vertex



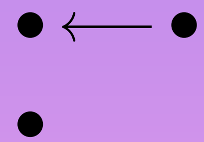
$M_{\mathcal{H}}$ cocones with $\mathcal{S}$-morphisms



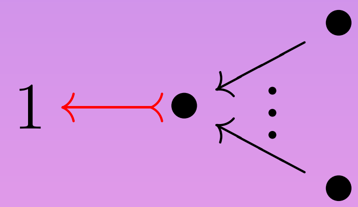
M_{Σ} cocones with one arrow



M_{Σ^+} cocones with 0 or 1 arrow



$M_{\mathcal{H}^-}$ cocones with vertex ≤ 1



The cocone classes $M_{\vec{\mathcal{X}}}$

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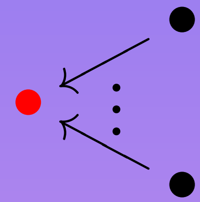
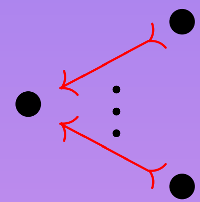
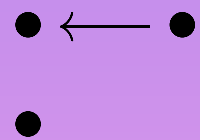


$M_{\mathcal{H}}$ cocones with $\mathcal{S}$-morphisms

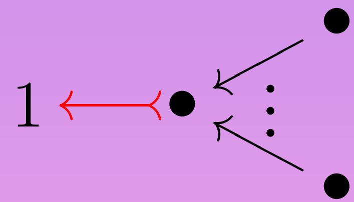


M_{Σ} cocones with one arrow

M_{Σ^+} cocones with 0 or 1 arrow



$M_{\mathcal{H}^-}$ cocones with vertex ≤ 1



As before, for composites $\vec{\mathcal{X}} = \mathcal{X}_1 \ldots \mathcal{X}_n$,

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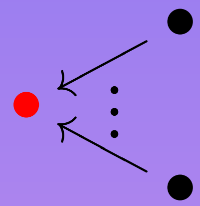
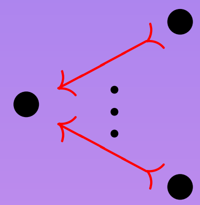
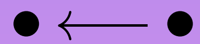
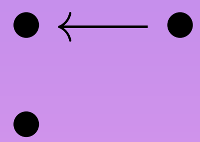
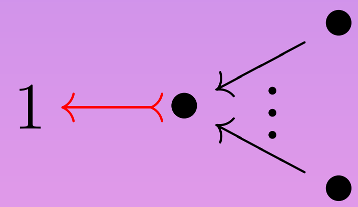
The cocone classes $M_{\vec{\mathcal{X}}}$

$M_{\mathcal{S}}$	cocones with injective vertex	
$M_{\mathcal{H}}$	cocones with $\mathcal{S}$-morphisms	
M_{Σ}	cocones with one arrow	
M_{Σ^+}	cocones with 0 or 1 arrow	
$M_{\mathcal{H}^-}$	cocones with vertex ≤ 1	

Also as before, $K_{\vec{\mathcal{X}}}\mathbf{V} = \{c \in M_{\vec{\mathcal{X}}} \mid \mathbf{V} \subseteq \mathbf{Proj}(\vec{\mathcal{X}})\}$.

An augmented slew

Let $\vec{\mathcal{X}}$ be a composite of $\mathcal{H}^-$, $\mathcal{S}$, $\mathcal{H}$, Σ and Σ^+ such that

- the operators occur in the order above;
- $\mathcal{H}$ occurs in $\vec{\mathcal{X}}$.

$$\mathbf{Proj}(K_{\vec{\mathcal{X}}}\mathbf{V}) = \vec{\mathcal{X}}\mathbf{V}$$

Upcoming topics

- What happened to coequations?

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- What happened to coequations?
- What is the formal dual to Birkhoff's completeness theorem?
- What is the analogue to Birkhoff's completeness theorem (and the corresponding theorem for conditional coequations)?