A Step Towards Deductive Completeness

Jesse Hughes

jesseh@cs.kun.nl

University of Nijmegen

Outline

I. A coequational language

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II. A coequational calculus

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III. $\mathrm{Ded}_G\, S$ is pre-complete

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Let $\Gamma : \mathcal{C} \to \mathcal{C}$ preserve $\mathcal{S}$-morphisms and suppose, further, that $U : \mathcal{C}_\Gamma \to \mathcal{C}$ has a right adjoint H. Then, we know that $\mathcal{C}_\Gamma$ satisfies the above conditions as well. Furthermore, U creates the factorization system in $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Gamma$ has enough cofree $\mathcal{S}$-injectives.

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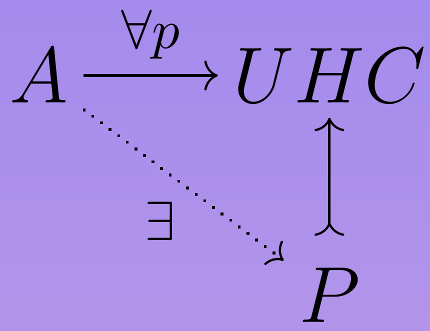
A coalgebra $\langle A,\, \alpha \rangle$ satisfies P iff for every $p : \langle A,\, \alpha \rangle \to HC$, $\mathsf{Im}(p) \leq HC$.



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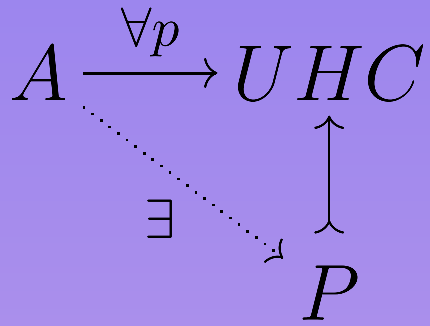

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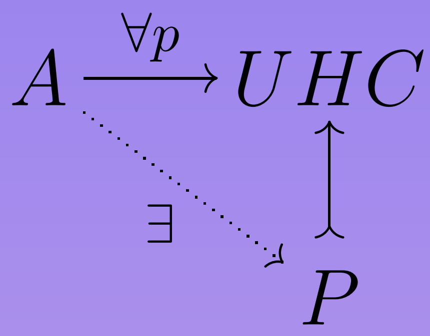

Hereafter, we further assume that $\mathcal{C}$ has all meets of $\mathcal{S}$-morphisms.
We denote the isomorphism classes of $\mathcal{S}$-morphisms $P \to C$ in $\mathcal{C}$ by $\mathsf{Sub}(C)$ – however, this notation is merely suggestive.

A coequational language

Fix a $\mathcal{S}$-injective $C \in \mathcal{C}$. We define a simple language $\mathcal{L}_{\mathsf{Coeq}}$ (properly, $\mathcal{L}^{C}_{\mathsf{Coeq}}$).

- For every P in $\mathsf{Sub}(UHC)$, we introduce an atomic proposition P in $\mathcal{L}_{\mathsf{Coeq}}$, i.e., $\mathsf{Sub}(UHC) \subseteq \mathcal{L}_{\mathsf{Coeq}}$.

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We define an interpretation $[\![-]\!]\colon \mathcal{L}_{\mathsf{Coeq}} \longrightarrow \mathsf{Sub}(UHC)$:

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Satisfaction

We say that a coalgebra $\langle A, \alpha \rangle$ satisfies a formula $\varphi \in \mathcal{L}_{\mathsf{Coeq}}$ (written $\langle A, \alpha \rangle \models \varphi$) just in case $\langle A, \alpha \rangle \models [\![\varphi]\!]$ in the sense of our previous talk.

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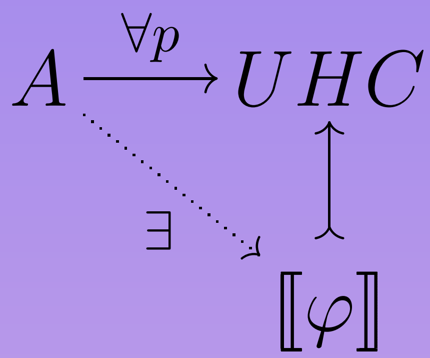
That is, $\langle A, \alpha \rangle \models \varphi$ just in case every homomorphism $\langle A, \alpha \rangle \to HC$ factors through the inclusion $[\![\varphi]\!] \rightarrowtail HC$.



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$$\begin{array}{ccc} A & \xrightarrow{\forall p} & UHC \\ & \overset{\exists}{\searrow} & \uparrow \\ & & [\![\varphi]\!] \end{array}$$

For a set $S \subseteq \mathcal{L}_{\mathsf{Coeq}}$, we say that $\langle A, \alpha\rangle \models S$ just in case $\langle A, \alpha\rangle \models \varphi$ for each $\varphi \in S$.

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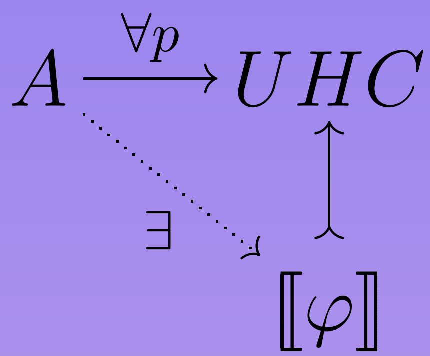

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We say that a collection $\mathbf{V} \subseteq \mathcal{C}_\Gamma$ satisfies S if each $\langle A, \alpha\rangle \in \mathbf{V}$ satisfies S.

(Pre-)complete sets of formulas

Recall our definition of *generating coequation* for a collection of coalgebras $\mathbf{V}$.

$\mathsf{Gen}\,\mathbf{V}$ satisfies the following fixed point description.

- $\mathbf{V} \models \mathsf{Gen}\,\mathbf{V}$;
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Call a set $S \subseteq \mathcal{L}_{\mathsf{Coeq}}$ of coequations over C *pre-complete* if there is a $\varphi \in S$ such that $[\![\varphi]\!] = \mathsf{Gen}\,\mathsf{Mod}(S)$.

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Call S *complete* if, for every φ such that $\mathsf{Mod}(S) \models \varphi$, we have $\varphi \in S$.

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We write $\varphi \vdash \psi$ just in case $[\![\varphi]\!] \vdash [\![\psi]\!]$, that is, just in case there is a morphism $[\![\varphi]\!] \rightarrow [\![\psi]\!]$ making the diagram below commute.

$$\begin{array}{ccc} [\![\varphi]\!] & \rightarrow & [\![\psi]\!] \\ & \searrow \quad \swarrow & \\ & UHC & \end{array}$$

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Call S *complete* if, for every φ such that $\mathsf{Mod}(S) \models \varphi$, we have $\varphi \in S$.
A pre-complete set S is complete just in case it is *upward-closed*, in the sense that if $\varphi \vdash \psi$ and $\varphi \in S$, then $\psi \in S$.

A sound rule

An inference rule $\dfrac{\varphi_1 \quad \ldots \quad \varphi_n}{\psi}$ is sound just in case, whenever $\langle A, \alpha \rangle \models \varphi_1, \ldots, \langle A, \alpha \rangle \models \varphi_n$, then $\langle A, \alpha \rangle \models \psi$.

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More generally, an (infinitary) inference rule $\dfrac{\{\varphi_i\}_{i\in I}}{\psi}$ is sound just in case, whenever $\langle A,\ \alpha\rangle \models \varphi_i$ for every $i \in I$, then $\langle A,\ \alpha\rangle \models \psi$.

A sound rule

More generally, an (infinitary) inference rule $\frac{\{\varphi_i\}_{i \in I}}{\psi}$ is sound just in case, whenever $\langle A, \alpha\rangle \models \varphi_i$ for every $i \in I$, then $\langle A, \alpha\rangle \models \psi$.

Theorem. *The rule* $\frac{\bigwedge \varphi_i}{\varphi_i} \bigwedge$*-E is sound.*

A sound rule

Theorem. $\bigwedge$*-E is sound.*

Proof. Suppose $\langle A, \alpha\rangle \models \bigwedge \varphi_i$ and $p : \langle A, \alpha\rangle \longrightarrow HC$. We must show that $\mathsf{Im}(p) \leq [\![\varphi_i]\!]$.

⋙

$$\begin{array}{ccc} A & \xrightarrow{p} & UHC \\ & & \big\uparrow \\ & & [\![\varphi_i]\!] \end{array}$$

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Theorem. $\bigwedge$*-E is sound.*

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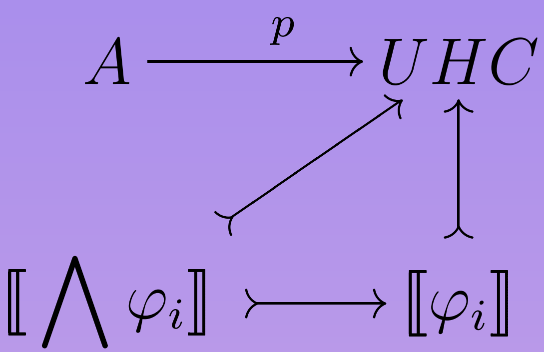

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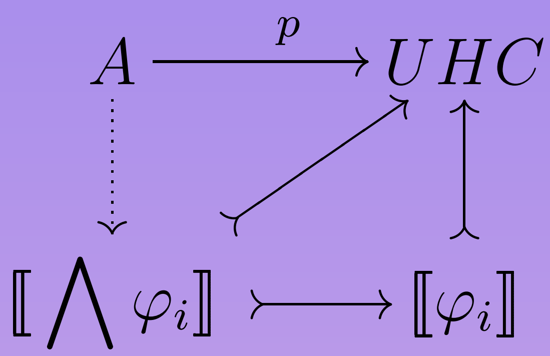

A sound rule

Theorem. *$\bigwedge$-E is sound.*

Proof. Suppose $\langle A, \alpha\rangle \models \bigwedge \varphi_i$ and $p : \langle A, \alpha\rangle \to HC$. We must show that $\mathsf{Im}(p) \le [\![\varphi_i]\!]$. But we know $\mathsf{Im}(p) \le [\![\bigwedge \varphi_i]\!] \le [\![\varphi_i]\!]$. □

This is a sound rule, but it's quite useless for our purposes.

A coequational calculus

The following rules are sound.

$$\frac{\{\varphi_i\}_{i\in I}}{\bigwedge \varphi_i} \bigwedge\text{-I}$$

If $\mathsf{Im}(p\!:\!\langle A,\,\alpha\rangle\!\to\! HC) \leq [\![\varphi_i]\!]$ for each $i \in I$, then $\mathsf{Im}(p) \leq \bigwedge [\![\varphi_i]\!]$.

A coequational calculus

The following rules are sound.

$$\frac{\{\varphi_i\}_{i\in I}}{\bigwedge \varphi_i} \bigwedge\text{-I} \qquad\qquad \frac{\varphi}{\Box\varphi} \Box\text{-I}$$

If $\mathsf{Im}(p\colon\langle A,\,\alpha\rangle\to HC) \leq [\![\varphi]\!]$, then $\mathsf{Im}(p) \leq \Box[\![\varphi]\!]$ (because $\mathsf{Im}(p)$ is a subcoalgebra contained in φ).

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$$\frac{\{\varphi_i\}_{i\in I}}{\bigwedge \varphi_i} \bigwedge\text{-I} \qquad \frac{\varphi}{\Box\varphi} \Box\text{-I}$$

$$\frac{\varphi}{\varphi(h(x))} \text{Subst}$$

Here, Subst applies for every Γ-homomorphism $h : HC \to HC$.

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Let $p : HC \to HC$ be given.

$$\mathsf{Im}(p) \leq h^*[\![\varphi]\!] \text{ iff } \exists_h \mathsf{Im}(p) \leq [\![\varphi]\!].$$

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Let $p\colon HC\to HC$ be given.

$$\mathsf{Im}(p) \leq h^*[\![\varphi]\!] \text{ iff } \mathsf{Im}(h\circ p) \leq [\![\varphi]\!].$$

Hence, if for every $q\colon HC\to HC$, $\mathsf{Im}(q) \leq [\![\varphi]\!]$, then $\mathsf{Im}(p) \leq h^*[\![\varphi]\!]$.

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Let's call this logic G (for pretty good logic).

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$$\frac{\{\varphi_i\}_{i \in I}}{\bigwedge \varphi_i} \bigwedge\text{-I} \qquad \frac{\varphi}{\Box\varphi} \Box\text{-I}$$

$$\frac{\varphi}{\varphi(h(x))} \text{Subst} \qquad \frac{\varphi \qquad \varphi \vdash \psi}{\psi} \text{DSR}$$

This is clearly a sound rule – if every map $\langle A, \alpha \rangle \to HC$ factors through $[\![\varphi]\!]$ and $[\![\varphi]\!] \leq [\![\psi]\!]$ then every such morphism also factors through $[\![\psi]\!]$.

A coequational calculus

The following rules are sound.

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However, it's not a rule we would generally like in our so-called logic, as it depends on the semantics of φ and ψ. Hence, we call it DSR for Damned Semantic Rule.

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We call the logic $G + \mathrm{DSR}$ a not-so-good logic, N.

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I. A coequational language

II. A coequational calculus

III. $\mathrm{Ded}_G\, S$ is pre-complete

IV. $\mathrm{Ded}_N\, S$ is complete

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A lemma

Lemma.

$$\boxslash[\![\varphi]\!] = \bigwedge\{h^*[\![\varphi]\!] \mid h : HC \longrightarrow HC\}.$$

In other terms,

$$\boxslash[\![\varphi]\!] = [\![\bigwedge\{\varphi(h(x)) \mid h : HC \longrightarrow HC\}]\!].$$

A lemma

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Proof. Recall $\boxslash[\![\varphi]\!] = \bigvee\{P \mid \forall h : HC \to HC \,.\, \exists_h P \leq [\![\varphi]\!]\}$.

$\supseteq$: It suffices to show that for all $k : HC \to HC$,

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$$\exists_k \boxslash [\![\varphi]\!] \leq [\![\varphi]\!].$$

But, $\boxslash [\![\varphi]\!]$ is invariant, so $\exists_k \boxslash [\![\varphi]\!] \leq \boxslash [\![\varphi]\!] \leq \varphi$.

G is pre-complete

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$$\frac{S}{\psi} \bigwedge\text{-I}$$

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Theorem. *Let* $S \subseteq \mathcal{L}_{\mathsf{Coeq}}$. *Then* $\mathsf{Ded}_G(S)$ *is pre-complete.*

Proof. So, we see that $S \vdash \Box \bigwedge \{\psi(h(x)) \mid h : HC \longrightarrow HC\}$. Now, by the lemma,

$$[\![\Box \bigwedge \{\psi(h(x)) \mid h : HC \longrightarrow HC\}]\!] = \Box \boxslash [\![\psi]\!],$$

and by the Invariance Theorem, $\Box \boxslash [\![\psi]\!] \leq [\![\varphi]\!]$. $\Box$

$\mathsf{Ded}_N\, S$ is complete.

Theorem. *Let* $S \subseteq \mathcal{L}_{\mathsf{Coeq}}$ *and let* $\mathsf{Ded}_N(S)$ *denote the deductive closure of* S *with respect to* N*. Then* $\mathsf{Ded}_N(S)$ *is complete.*

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Proof. Recall that N is $G + \mathrm{DSR}$, where DSR is the rule

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Proof. Recall that N is $G + \text{DSR}$, where DSR is the rule

$$\frac{\varphi \qquad \varphi \vdash \psi}{\psi}$$

Hence, $\mathsf{Ded}_N(S)$ is the upward closure of $\mathsf{Ded}_G(S)$, which is pre-complete. □

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An implicational language

Define $\mathcal{L}_{\mathsf{Imp}} = \{\varphi \Rightarrow \psi \mid \varphi, \psi \in \mathcal{L}_{\mathsf{Coeq}}\}$.
Say that $\langle A,\, \alpha\rangle \models \varphi \Rightarrow \psi$ just in case, for every $p\colon\langle A,\, \alpha\rangle \to HC$ such that $\mathsf{Im}(p) \leq [\![\varphi]\!]$, also $\mathsf{Im}(p) \leq [\![\psi]\!]$.

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$$\begin{array}{ccc} A & \xrightarrow{p} & UHC \\ & \searrow & \big\uparrow \\ & & [\![\varphi]\!] \end{array} \quad \Rightarrow \quad \begin{array}{ccc} A & \xrightarrow{p} & UHC \\ & \searrow & \big\uparrow \\ & & [\![\psi]\!] \end{array}$$

Reminder: This is not the same as ($\langle A,\ \alpha\rangle \not\models \varphi$ or $\langle A,\ \alpha\rangle \models \psi$). That would be true if either there is some p such that $\mathsf{Im}(p) \not\leq [\![\varphi]\!]$ or for all p, $\mathsf{Im}(p) \leq [\![\psi]\!]$.

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This is also not the same as $\langle A,\, \alpha\rangle \models \neg\varphi \vee \psi$ (if $\mathsf{Sub}(UHC)$ is a Heyting algebra).

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Note:

$$\langle A,\, \alpha\rangle \models \varphi \text{ iff } \langle A,\, \alpha\rangle \models \top \Rightarrow \varphi,$$

where $\top = (HC{=}HC)$.

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Let's call this logic G^i, again because it seems a reasonably good logic.

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$$\frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \vartheta}{\varphi \Rightarrow \vartheta} \text{Cut} \qquad \frac{\varphi \Rightarrow \psi \qquad \psi \vdash \vartheta}{\varphi \Rightarrow \vartheta} \text{DSR}$$

It's that damned semantic rule again. Let's call this N^i for not so good implicational logic.

A couple of handy operators

We say that a coequation φ is S-*minimal* just in case, whenever $S \models \varphi \Rightarrow \psi$, then $\varphi \vdash \psi$.

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Given $S \subseteq \mathcal{L}_{\mathsf{Imp}}$, define two operators $\mathsf{Sub}(UHC) \rightarrow \mathsf{Sub}(UHC)$:

$$\mathbf{cons}_S\,\varphi = \bigwedge \{\psi \mid \varphi \Rightarrow \psi \in S\}$$

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$\mathbf{cons}_S\, \varphi$ is the meet of the consequents of φ in S.

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Lemma. $\mathbf{ent}_S(\varphi)$ *is* S-minimal, *and hence is the greatest* S-minimal *subobject below* φ*.*

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Lemma.

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Lemma.

$$\mathbf{ent}_S\,\varphi = \bigwedge\{\psi \mid \mathsf{Mod}(S) \models \varphi \Rightarrow \psi\}$$

So, S is pre-complete iff for every φ, we have $\varphi \Rightarrow \mathbf{ent}_S\,\varphi \in S$. Our goal is to show that $\mathsf{Ded}_{G^i}\,S$ contains $\varphi \Rightarrow \mathbf{ent}_S\,\varphi$.

Definition of EIEIO

Call an operator $\boxdot : \mathcal{L}_{\mathsf{Imp}} \to \mathcal{L}_{\mathsf{Imp}}$ an *endomorphism-invariant interior operator* (**EIEIO**) just in case it satisfies the following axioms.

$$\frac{}{\boxdot\varphi \vdash \Box \boxdot \varphi}\ \mathrm{S/C}$$

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$$\frac{h : HC \longrightarrow HC}{\exists x(\boxdot\varphi(x) \wedge h(x) = y) \vdash \boxdot(\exists x(\varphi(x) \wedge h(x) = y))}\ \text{FEI}$$

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In other words, an operator $\boxdot$ is **EIEIO** just in case

- $\boxdot$ is a comonad (deflationary, idempotent, monotone);
- $\boxdot$ is *fully endomorphism invariant* – for all $h : HC \rightarrow HC$, $\exists x(\boxdot\varphi(x) \wedge h(x) = y) \vdash \boxdot(\exists x(\varphi(x) \wedge h(x) = y))$.

$\mathsf{Ded}_{Gi}\, S$ is pre-complete.

Lemma. $\mathbf{ent}_S$ *is the greatest* **EIEIO** *suboperator of* $\Box \circ \mathbf{cons}_S$. *That is,* $\mathbf{ent}_S \leq \Box \circ \mathbf{cons}_S$ *and for every* $\boxdot : \mathsf{Sub}(UHC) \to \mathsf{Sub}(UHC)$ *in* **EIEIO** *such that* $\boxdot \leq \Box \circ \mathbf{cons}_S$*, also* $\boxdot \leq \mathbf{ent}_S$.

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Lemma. *If S is deductively closed, then* $\mathbf{cons}_S$ *is an* **EIEIO**. *In other words,* $\mathbf{cons}_{\mathsf{Ded}_{Gi}\, S}$ *is an* **EIEIO**.

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Lemma. *If S is deductively closed, then* $\mathbf{cons}_S$ *is an* **EIEIO**. *In other words,* $\mathbf{cons}_{\mathsf{Ded}_{Gi}\, S}$ *is an* **EIEIO**.

Corollary. $\mathbf{cons}_{\mathsf{Ded}_{Gi}\, S} = \mathbf{ent}_S$.

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Proof. $\mathbf{cons}_{\mathsf{Ded}_{Gi}\,S}$ is an **EIEIO** and a suboperator of $\Box \circ \mathbf{cons}_S$. $\ggg$

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Corollary. $\mathbf{cons}_{\mathsf{Ded}_{Gi}\, S} = \mathbf{ent}_S$.

Proof. $\mathbf{cons}_{\mathsf{Ded}_{Gi}\, S}$ is an **EIEIO** and a suboperator of $\Box \circ \mathbf{cons}_S$. Hence, $\mathbf{cons}_{\mathsf{Ded}_{Gi}\, S} \leq \mathbf{ent}_S$. $\ggg$

$\mathsf{Ded}_{G^i}\, S$ is pre-complete.

Corollary. $\mathbf{cons}_{\mathsf{Ded}_{G^i}\, S} = \mathbf{ent}_S$.

Proof. $\mathbf{cons}_{\mathsf{Ded}_{G^i}\, S}$ is an **EIEIO** and a suboperator of $\Box \circ \mathbf{cons}_S$. Hence, $\mathbf{cons}_{\mathsf{Ded}_{G^i}\, S} \leq \mathbf{ent}_S$. The other inclusion follows from the fact that G^i is sound. $\Box$

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Proof. It suffices to show that $\mathsf{Ded}_{G^i}\, S$ contains $\varphi \Rightarrow \mathbf{ent}_S\, \varphi$ for each $\varphi \in \mathcal{L}_{\mathsf{Coeq}}$. $\ggg$

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Theorem. $\mathsf{Ded}_{G^i}\, S$ *is pre-complete.*

Proof. It suffices to show that $\mathsf{Ded}_{G^i}\, S$ contains $\varphi \Rightarrow \mathbf{ent}_S\, \varphi$ for each $\varphi \in \mathcal{L}_{\mathsf{Coeq}}$. Thus, it suffices to show that $\mathsf{Ded}_{G^i}\, S$ contains each $\varphi \Rightarrow \mathbf{cons}_{\mathsf{Ded}_{G^i}\, S}\, \varphi$.

$\ggg$

$\mathsf{Ded}_{G^i}\,S$ is pre-complete.

Theorem. $\mathsf{Ded}_{G^i}\,S$ *is pre-complete.*

Proof. It suffices to show that $\mathsf{Ded}_{G^i}\,S$ contains $\varphi \Rightarrow \mathbf{ent}_S\,\varphi$ for each $\varphi \in \mathcal{L}_{\mathsf{Coeq}}$. Thus, it suffices to show that $\mathsf{Ded}_{G^i}\,S$ contains each $\varphi \Rightarrow \mathbf{cons}_{\mathsf{Ded}_{G^i}\,S}\,\varphi$. This is clear, since

$$\mathbf{cons}_{\mathsf{Ded}_{G^i}\,S}\,\varphi = \bigwedge\{\psi \mid \varphi \Rightarrow \psi \in \mathsf{Ded}_{G^i}\,S\}$$

and $\mathsf{Ded}_{G^i}\,S$ is closed under the rule $\dfrac{\{\varphi \Rightarrow \psi_i\}_{i\in I}}{\varphi \Rightarrow \bigwedge \psi_i}\ \bigwedge\text{-I}$

$\square$

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By the previous argument, we see that $\mathsf{Ded}_{N^i}\, S$ is pre-complete. Clearly, it is also upward-closed and hence complete. □

Outline

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I. A coequational language
II. A coequational calculus
III. $\mathrm{Ded}_G\, S$ is pre-complete
IV. $\mathrm{Ded}_N\, S$ is complete
V. An implicational language
VI. An implicational calculus
VII. $\mathrm{Ded}_{G^i}\, S$ is pre-complete
VIII. $\mathrm{Ded}_{N^i}\, S$ is complete

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- An example of reasoning with one of these logics – is that even plausible?