Admissible Digit Sets

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Outline

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1. Digit sets
   - Binary representation
   - Möbius maps and digit sets
   - The Stern-Brocot representation
2. Admissibility
   - Admissible digit sets
   - The homographic algorithm

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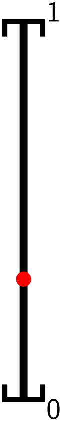
The standard binary representation of $[0, 1]$.



Think of binary representations in $[0, 1]$, like

$$0.010100010\ldots$$

$$\{0, 1\}^\omega \longrightarrow [0, 1]$$

$$x_1 x_2 x_3 \ldots \longmapsto \sum_{i=0}^{\infty} x_i \cdot 2^{-i}$$

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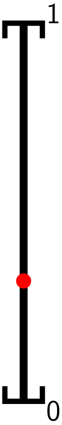

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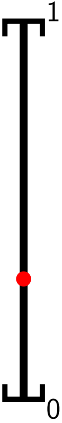

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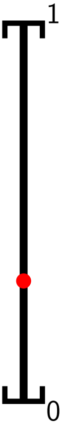

Think: receiving one bit at a time.

Each bit restricts the set of possibilities.

- With 0 bits, x could be anything in $[0, 1]$.
- When we see "0.0", the options are reduced.
- "0.01" reduces them further.

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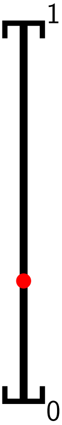

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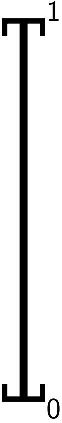

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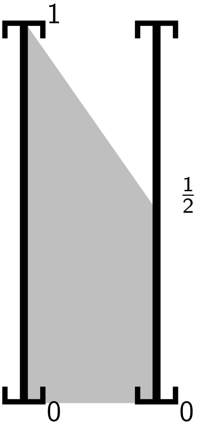

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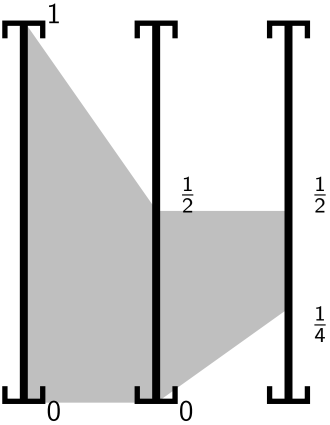

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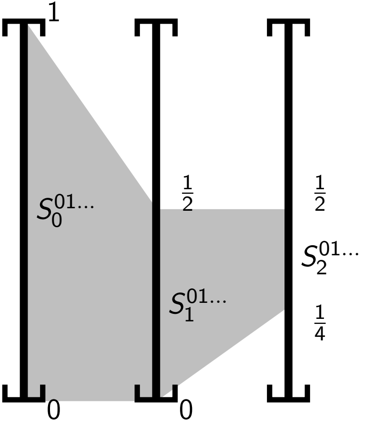

$$\vec{x} = x_1\, x_2\, x_3\ \ldots$$

$$S_0^{\vec{x}} \supsetneq S_1^{\vec{x}} \supsetneq S_2^{\vec{x}} \supsetneq S_3^{\vec{x}} \supsetneq \ldots$$

Some features:

- $\bigcap S_i^{\vec{x}}$ is a singleton.
- For each x, there is a sequence $\vec{x}$ such that $\bigcap S_i^{\vec{x}} = \{x\}$.

Each sequence represents some x and each x is represented.

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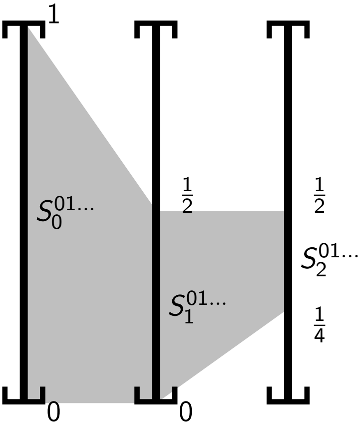

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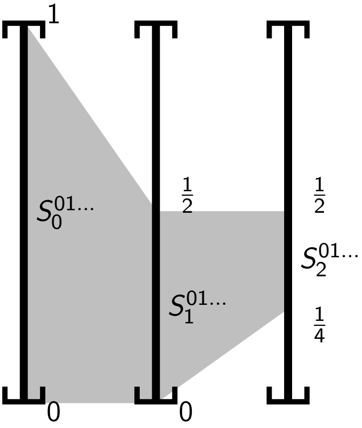

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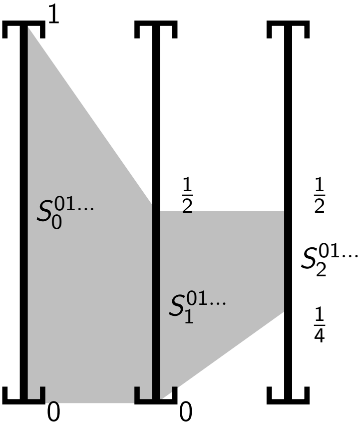

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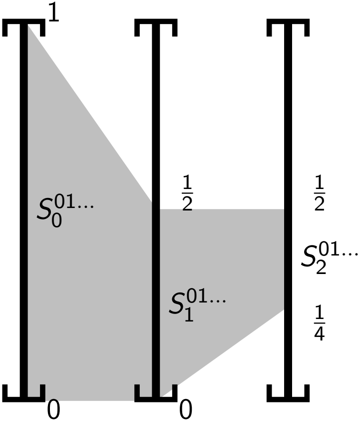

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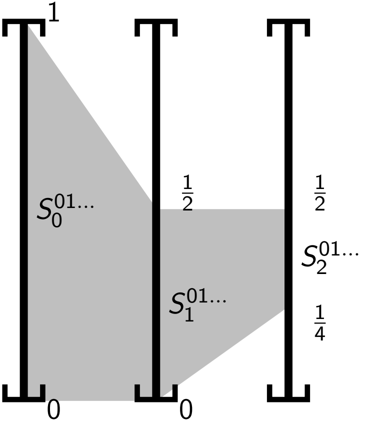

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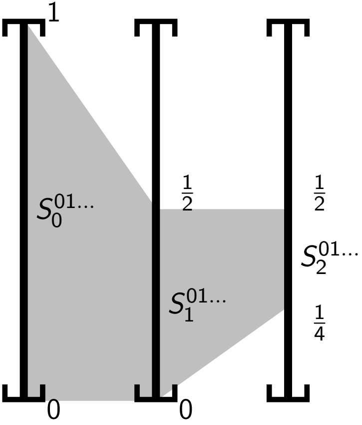

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$$S_i^{\vec{x}} = \phi_{x_0} \circ \ldots \circ \phi_{x_i}([0,1])$$
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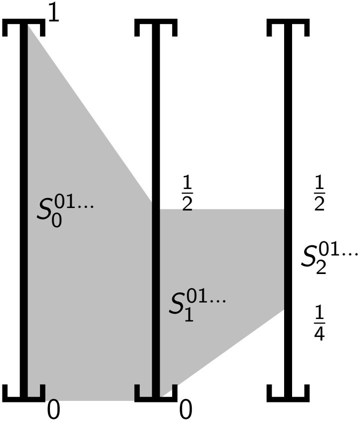

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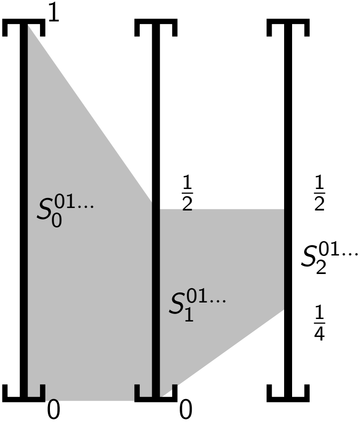

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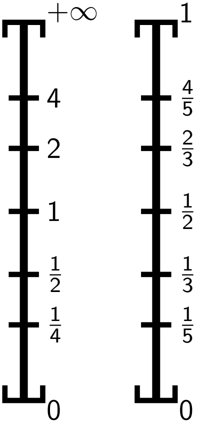

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The choice is arbitrary.

Squint and you can't tell the difference.

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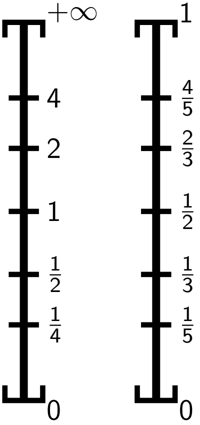

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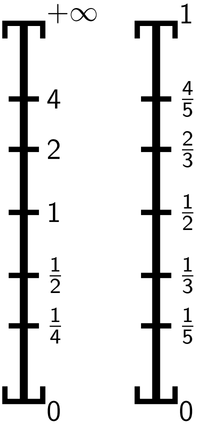

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Möbius maps

Recall $\phi_0(x) = \frac{x}{2}, \qquad \phi_1(x) = \frac{x+1}{2}$.

Möbius map: a function

$$A(x) = \frac{ax+b}{cx+d}$$

where $a, b, c, d \in \mathbb{R}$.

We are interested in Möbius maps that are

- strictly increasing,
- refining $(A : [0, +\infty] \to [0, +\infty])$.

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Digit sets

Möbius maps are our *digits*.

Let $\Phi = \{\phi_0, \ldots, \phi_k\}$ be a set of Möbius maps.

A sequence $\vec{x} = \phi_{i_0}\, \phi_{i_1}\, \phi_{i_2}\, \ldots$ *represents* x if

$$\bigcap_{n=0}^{\infty} \underbrace{\phi_{i_0} \circ \phi_{i_1} \circ \ldots \circ \phi_{i_n}([0, +\infty])}_{S_n^{\vec{x}}} = \{x\}.$$

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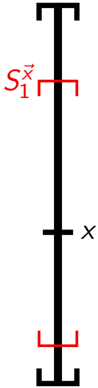

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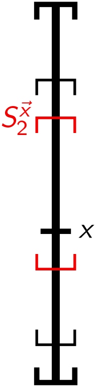

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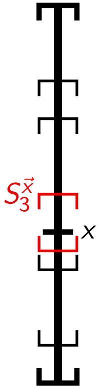

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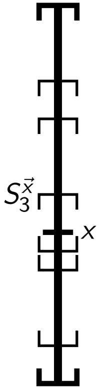

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Good digit sets

Φ is a *good digit set* if

1. Loosely: $\bigcap S_i^{\vec{x}}$ is always a singleton.
2. The sets $\phi_i([0,+\infty])$ cover $[0,+\infty]$.

Theorem

Good digit sets are digit sets.

Good digit sets yield a *total representation*, i.e. $\Phi^\omega \to [0,+\infty]$ is

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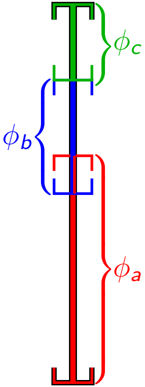

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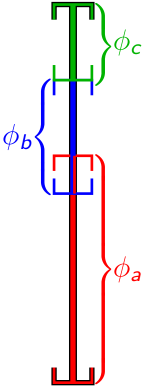

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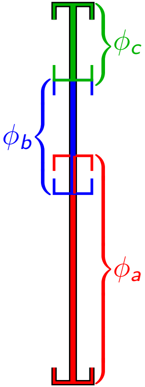

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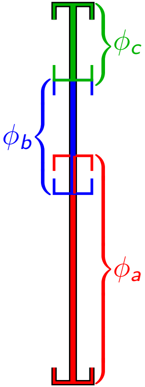

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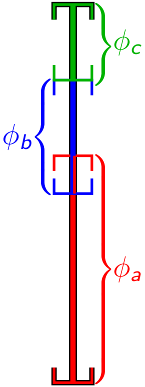

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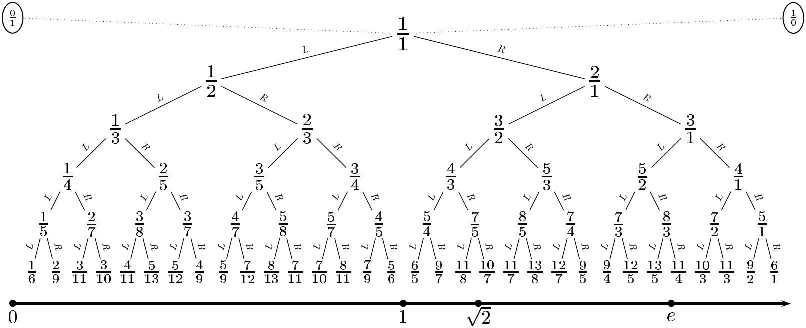

How to make the tree:

If my parents are $\frac{a}{b}$ and $\frac{c}{d}$, then I am $\frac{a+c}{b+d}$.

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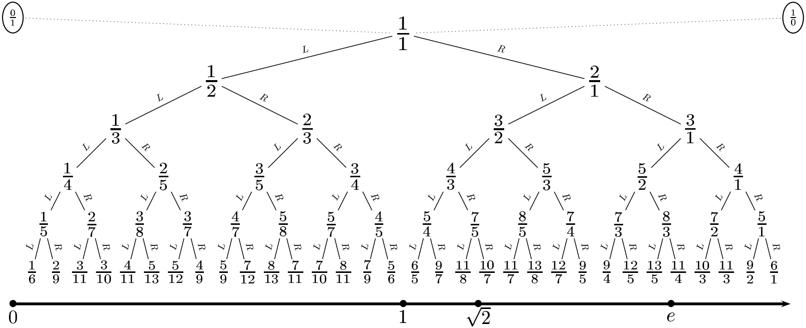
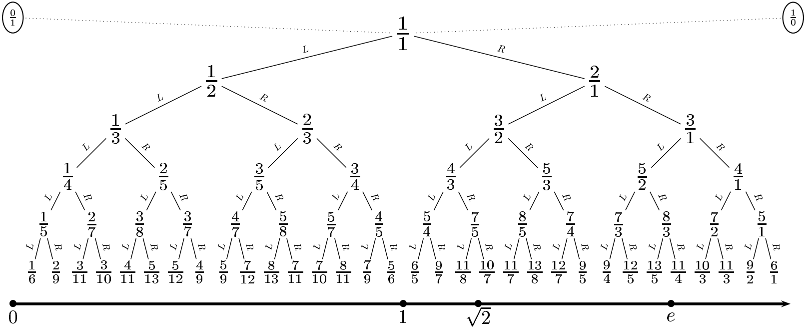


The Stern-Brocot representation maps finite sequences of $\{L, R\}$ to rationals.

The Stern-Brocot representation is a digit set



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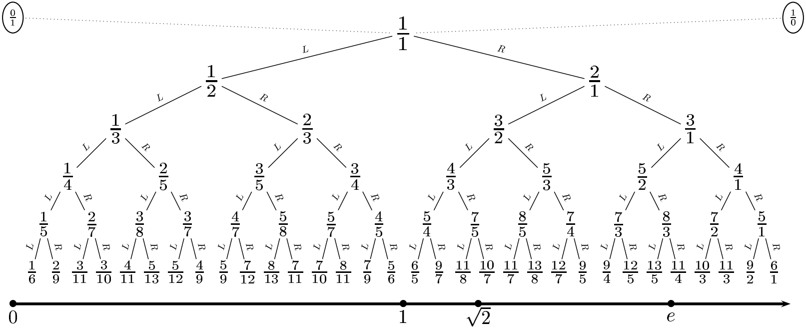

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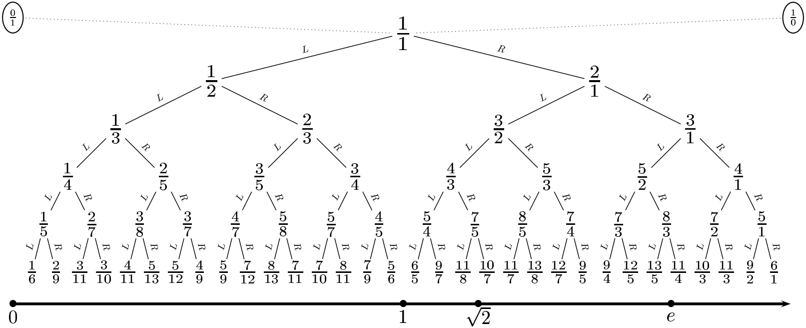

$$L \ldots \in [0, 1]$$

$$LR \ldots \in [\frac{1}{2}, 1]$$

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A nested sequence of sets $S_i^{\vec{x}}$.

Each $S_i^{\vec{x}}$ is bounded by the parents of $x_1\, x_2\, \ldots\, x_i$.

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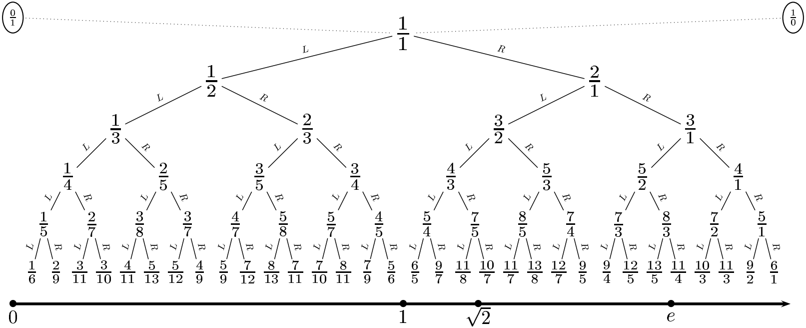

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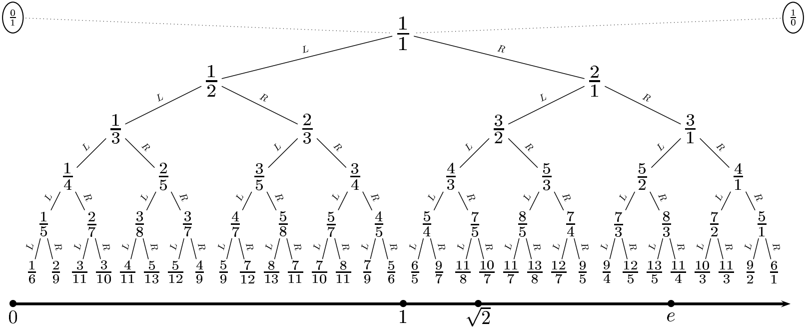

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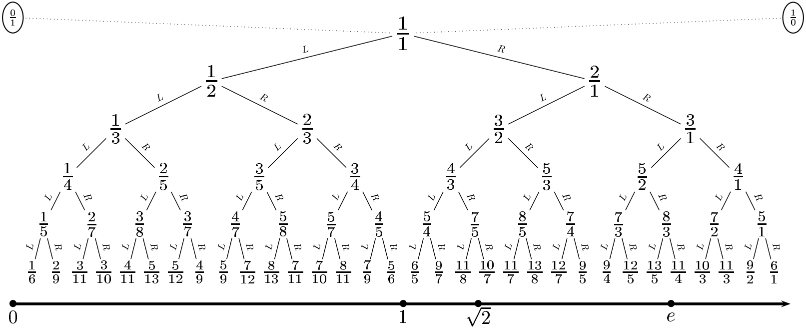

$$\phi_L(x) = \frac{x}{x+1} \qquad \phi_R(x) = x + 1$$

$\{\phi_L, \phi_R\}$ is a good digit set.

Outline

1. Digit sets
   - Binary representation
   - Möbius maps and digit sets
   - The Stern-Brocot representation
2. Admissibility
   - Admissible digit sets
   - The homographic algorithm

Φ-Computability

Let Φ be a good digit set.

$f : [0, +\infty] \to [0, +\infty]$ is Φ-computable iff f has a continuous Φ^ω lifting.

$$\begin{array}{ccc} \Phi^\omega & \overset{f^\sharp}{\dashrightarrow} & \Phi^\omega \\ \downarrow & & \downarrow \\ [0, +\infty] & \underset{f}{\longrightarrow} & [0, +\infty] \end{array}$$

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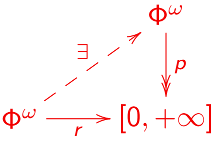

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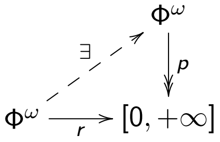

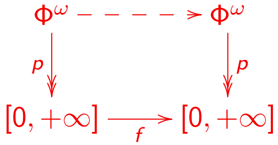

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Φ is an *admissible digit set* (ADS) if

1. Loosely: $\bigcap S_i^{\vec{x}}$ is always a singleton.
2. The sets $\phi_i((0, +\infty))$ cover $(0, +\infty)$.

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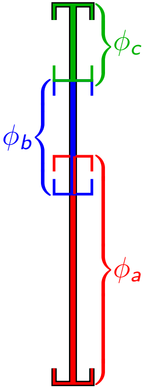

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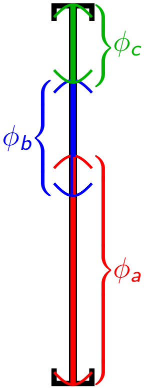

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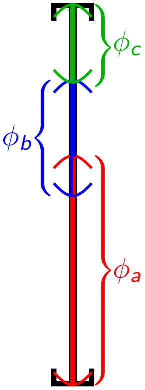

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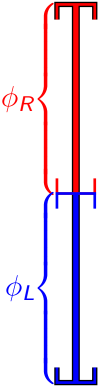

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$$\phi_L([0,+\infty]) = [0,1]$$
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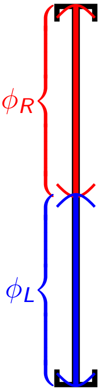

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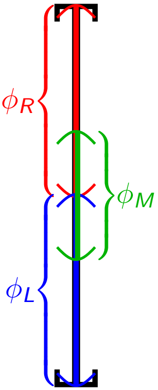

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Aim: Construct an algorithm $H(A, -)$ computing Möbius maps A.

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Any continuous $f : [0, +\infty] \to [0, +\infty]$ lifts to Φ^ω.

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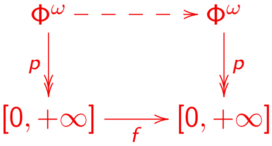

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An algorithm for computing Möbius maps

Let $\mathbb{M}$ be the set of refining Möbius maps.

We explicitly defined $H : \mathbb{M} \times \Phi^\omega \to \Phi^\omega$ so that

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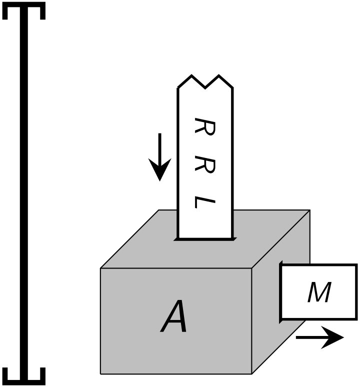

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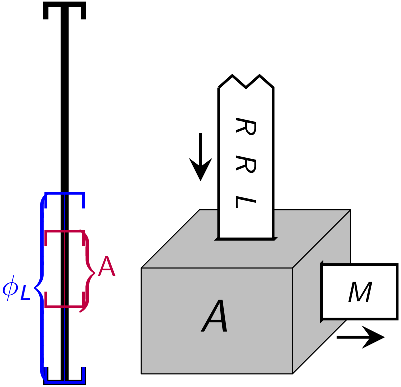

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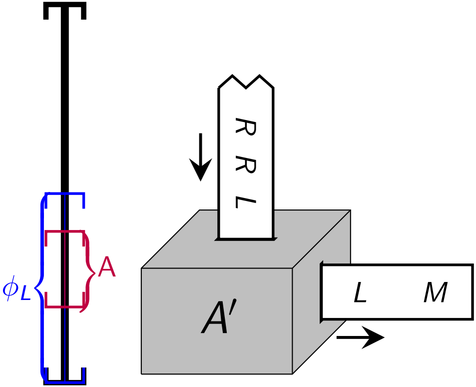

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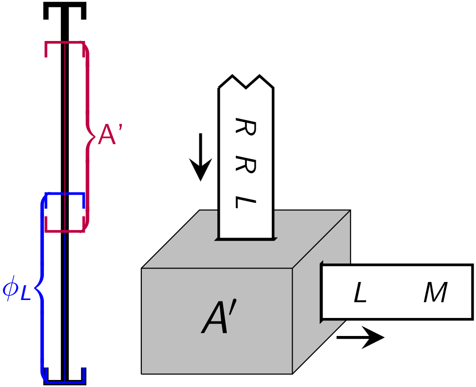

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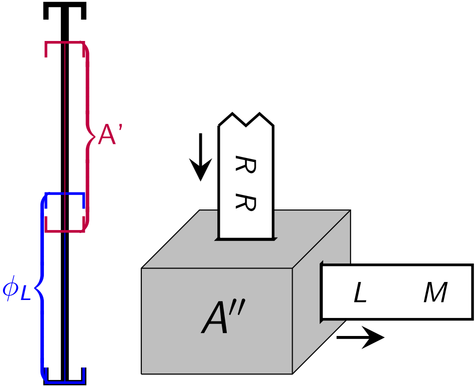

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- Aim: investigate representations via Möbius maps
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Möbius maps and matrices

$$A(x) = \frac{ax+b}{cx+d}$$

Same thing: A matrix $M_A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $x, y \in [0, +\infty)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \\ +\infty & \text{else.} \end{cases}$$

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Composition of Möbius maps is the same as multiplication of matrices.

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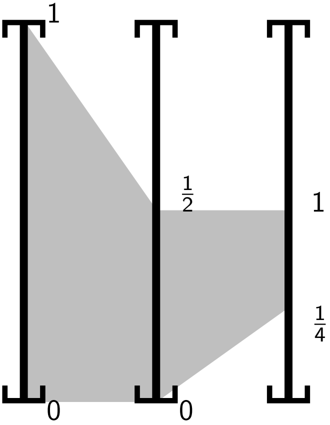
Translating ϕ_0, ϕ_1 to $[0, +\infty]$



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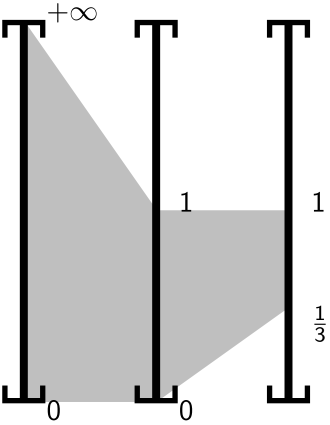

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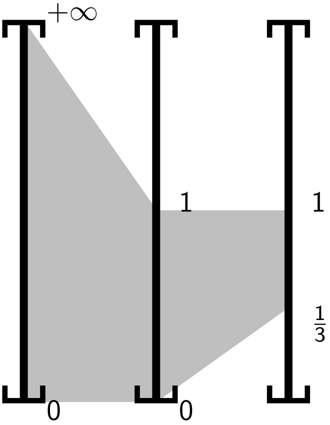

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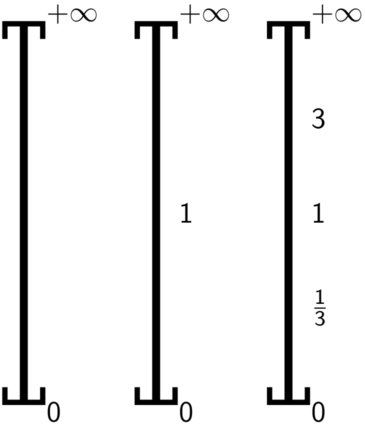
$$[0, 1] = \phi_0([0, +\infty])$$

$$[\frac{1}{3}, 1] = \phi_0 \circ \phi_1([0, +\infty])$$

$$\frac{1}{\pi - 1} \text{ “} = \text{” } .010100010\ldots$$

Good digit sets have shrinking diameters.



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We use this metric to measure the "shrinking" of $\phi_{i_1}\phi_{i_2} \ldots \phi_{i_n}([0, +\infty])$.

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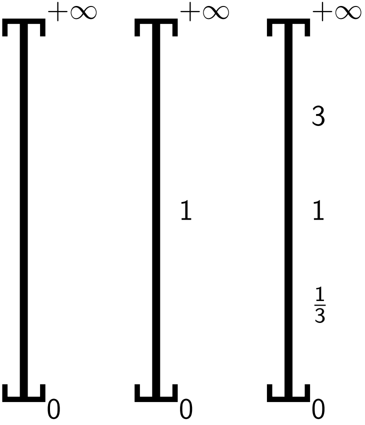

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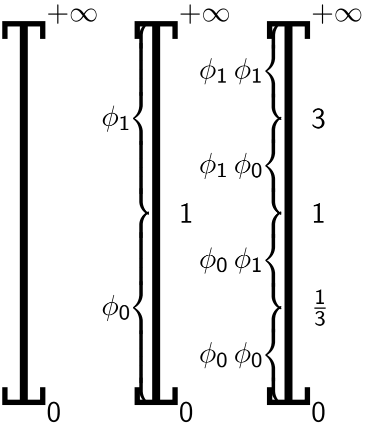

$\mathcal{B}(\Phi, n)$ measures the maximum diameter for n-length sequences.

$$\mathcal{B}(\Phi, 0) = 1$$
$$\mathcal{B}(\Phi, 1) = \frac{1}{2}$$
$$\mathcal{B}(\Phi, 2) = \frac{1}{4}$$

Good: $\lim_{j \to \infty} \mathcal{B}(\Phi, j) = 0$

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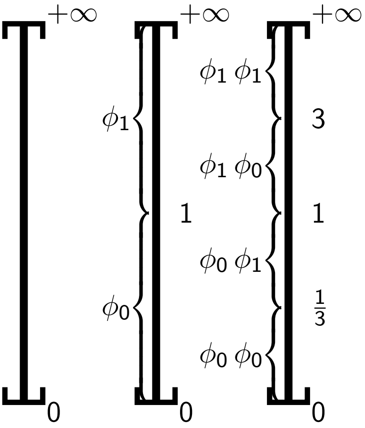

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$$\mathcal{B}(\Phi, 1) = \frac{1}{2}$$

$$\mathcal{B}(\Phi, 2) = \frac{1}{4}$$

Good: $\lim_{j\to\infty} \mathcal{B}(\Phi, j) = 0$

The rough idea behind the algorithm

When $A(x) \in \phi_j((0, +\infty))$ no matter what x is, output the digit ϕ_j.

Otherwise, absorb a digit from x to refine our calculation.

Define $A \sqsubseteq \phi_j \Leftrightarrow A([0, +\infty]) \subseteq \phi_j([0, +\infty])$.

$$H(A, \phi_{i_1}\phi_{i_2}\ldots) := \begin{cases} \phi_0\ H(\phi_0^{-1} \circ A, \phi_{i_1}\phi_{i_2}\ldots) & \textbf{if } A \sqsubseteq \phi_0 \\ \phi_1\ H(\phi_1^{-1} \circ A, \phi_{i_1}\phi_{i_2}\ldots) & \textbf{else if } A \sqsubseteq \phi_1 \\ \vdots & \\ \phi_k\ H(\phi_k^{-1} \circ A, \phi_{i_1}\phi_{i_2}\ldots) & \textbf{else if } A \sqsubseteq \phi_k \\ H(A \circ \phi_i, \phi_{i_2}\phi_{i_3}\ldots) & \textbf{otherwise}. \end{cases}$$

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