Admissible Digit Sets and a Modified Stern–Brocot Representation

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Abstract

We examine a special case of admissible representations of the closed interval, namely those which arise via sequences of a finite number of Möbius transformations. We regard certain sets of Möbius transformations as a generalized notion of digits and introduce sufficient conditions that such a "digit set" yields an admissible representation of $[0, +\infty]$. Furthermore we establish the productivity and correctness of the homographic algorithm for such "admissible" digit sets. In the second part of the paper we discuss representation of positive real numbers based on the Stern–Brocot tree. We show how we can modify the usual Stern–Brocot representation to yield a ternary admissible digit set.

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1 Introduction

We investigate the role of *redundancy* in real number representations, especially as it pertains to computability of real-valued functions. In particular, we are concerned with redundancy in intensional approaches to exact arithmetic in which real numbers are represented by the data-type of infinite sequences (synonymously, *streams*). In the functional programming community, it has long been known that such representations yield "useful" algorithms only if each real number has more than one representation [28, 3]. In particular, redundancy ensures that the relevant lazy algorithms on streams are *productive* [5, 23]. Relatedly, Type Two Effectivity (TTE) provides a powerful framework to study the computational properties of real number representations, including those given by the Cantor or Baire spaces, [26, 27], and plays a key role in our development.

The motivation for our work comes from an ongoing project to formalize the algorithms of exact real arithmetic and verify their correctness in the *Coq* [24] theorem prover. In the previous phases of this project, the algorithms for exact rational arithmetic were verified [19]. While trying to adapt the formalization for the real numbers, it became clear that one must carefully analyze the topological properties of the representation and their relation to productivity of the algorithms. This resulted in a new formulation of *digit set*, which suffices to show that the resulting representation is *admissible* in the sense of TTE (cf. Section 3). This also yielded fairly general methods of proving the productivity and correctness of the algorithms on streams (cf. Section 4). Because of its type theoretic nature, this generic method is easy to formalize inside a theorem

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prover. However in the present paper we do not mention the issues specific to the formalization in a theorem prover.

Computations on continued fractions have led to many of the approaches to exact arithmetic. The main idea of these approaches were explicit in the early works of Gosper [8] and Raney [22] and were later further developed, generalized and implemented in various contexts [25, 16, 17, 21, 7, 20, 15, 6, 14, 9]. In short in these approaches, a real number is represented by a stream of suitably chosen maps. This stream is interpreted as a limit of the composition of the maps applied to a base interval. The maps are considered as the digits representing the real number. Our approach is a part of this tradition, motivated by the practical considerations of formal verification.

A basic difference in the approaches in the literature are the conditions restraining the set of digit maps. Boehm et al. [3] present the notion of *interval representation* and justify (via recursion theory) that for a subclass of these representations, algorithms for addition and subtraction are total. Nielsen and Kornerup [17] present a similar general framework based on axiomatization of a *digit serial number representation*, in which the digit maps are contractive on real intervals and the limit of the compositions are singletons. Examples of digit serial number representation include ordinary radix (eg. decimal, binary) representation, continued fraction representation and the more general LFT (Linear fractional transformation) representation. LFT representations (in which the digit maps are hyperbolic Möbius maps, corresponding via group conjugations to radix representation) were developed and implemented by Edalat and Potts [21, 20, 6] and Heckmann [11, 12]. These restrictions give rise to elegant algorithms for transcendental functions. Konečný [14] restricts the set of maps to *d-contractions*, which are twice-differentiable functions with a unique fixpoint and positive derivative. This leads to the notion of *IFS-representation*, which includes both radix and LFT representation.

A common feature of each of these approaches is that each real has multiple representations, i.e., the representations are redundant. This seems a common feature of computationally useful representations, but the expression "computationally useful" has different intended meanings, partly due to the applications in which the developers are interested. We prefer the elegant notion of computability on streams provided by TTE and hence show that our digit sets yield admissible representations. Moreover, we explicitly show that the usual homographic algorithms are productive for these representations.

The present paper has two distinct parts: (1) a general presentation of (admissible) digit sets together with a confirmation that the homographic algorithms are productive (Sections 2 – 4) and (2) a modified Stern-Brocot encoding, with proof that it is an admissible digit set (Sections 5 – 6). In Section 2 we introduce the basics of the intensional approach to representing real numbers for exact real arithmetic. In Section 3 we focus on *admissible representations* in the context of Möbius maps. We introduce the notion of admissible digit set and prove that our criteria for an admissible digit set indeed leads to an admissible representation. Theorem 3.9 is the main result of this section and is based on a result by Brattka and Hertling [4]. In Section 4 we show that the so-called refining Möbius maps are induced by productive functions on streams over admissible digit sets. Productivity for functions on infinite objects is the dual notion to termination for recursive functions and is the type theoretic counterpart of continuity.

In Section 5 we present the binary Stern–Brocot representation, and in Section 6 we modify this by adding an additional digit to yield an admissible representation.

2 Real Numbers and Möbius Transformations

In this section we present some definitions from the theory of Möbius transformations which we will use in the rest of the paper. A *Möbius map* is a map

$$x \longmapsto \frac{ax+b}{cx+d},$$

where $a, b, c, d \in \mathbb{R}$. *Nonsingular* Möbius maps are those for which we have $ad - bc \neq 0$, i.e., those which are strictly monotone and hence injective. Every 2×2 matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ denotes a

Möbius map given by $\phi_A(x) = \frac{ax+b}{cx+d}$. In this sense, nonsingular Möbius maps correspond to 2×2 matrices with a nonzero determinant (where a, b, c, d have no common divisor). In this paper we will identify a matrix A with its corresponding Möbius map, for example we write $A(x)$ for $x \in \mathbb{R}$. In particular we write $A(\infty)$ for $\lim_{x \to +\infty} A(x)$. It is clear that for every A either $A(\infty) = \frac{a}{c}$ or $A(\infty) = \infty$. Moreover, if A is nonsingular and $[x, y]$ is a closed real interval, then $A([x, y])$ denotes the image of $[x, y]$ under A, namely $[A(x), A(y)]$ if A is increasing and $[A(y), A(x)]$ else.

We denote the set of positive reals by $\mathbb{R}^+$ and we define $\mathbb{R}^\star = \mathbb{R}^+ \cup \{0, +\infty\}$. In the rest of the paper we consider the set $\mathbb{R}^\star$ as our *base interval.* However all the results of the paper apply to any other compact interval.

A *refining* Möbius map is a nonsingular Möbius map that maps $\mathbb{R}^\star$ to itself. We denote the set of all refining Möbius maps by $\mathbb{M}$. The matrices corresponding to refining Möbius maps form a group with matrix multiplication. This matrix multiplication corresponds to composition of the corresponding Möbius maps. If we consider $\mathbb{R}^\star$ as the interval $[0, +\infty]$, then a map ϕ is refining iff $\phi(\mathbb{R}^\star) \subseteq \mathbb{R}^\star$.

We define $\Phi^{<\omega}$ (resp. Φ^ω) to be the set of all finite sequences (resp. all streams) of elements taken from $\Phi \subseteq \mathbb{M}$. We write ϕ for elements of Φ, and let $\sigma, \tau, \ldots$ range over $\Phi^{<\omega}$, while $\alpha, \beta, \ldots$ range over Φ^ω and write $(\alpha)_j$ for the jth position of α. We write sequences $\phi_0\phi_1 \cdots \phi_k$ and streams similarly. Let $\phi_0\phi_1 \cdots$ be an infinite stream of Möbius maps. We define their *infinite composition* to be

$$\bigcap_{i=0}^{\infty} \phi_0 \circ \ldots \circ \phi_i(\mathbb{R}^\star).$$

It is easy to prove by induction that, if each ϕ_k is refining, then this is a nested intersection of closed intervals, and hence non-empty. If the intersection is a singleton $\{x\}$, then we say that x is *represented* by the infinite composition $\bigcap_{i=0}^{\infty} \phi_0 \circ \ldots \circ \phi_i(\mathbb{R}^\star)$.

Definition 2.1 A finite set Φ of Möbius maps is a *digit set* if each element x of $\mathbb{R}^\star$ is represented by some infinite composition of elements of Φ, i.e.,,

$$x = \bigcap_{i=0}^{\infty} \phi_0 \circ \ldots \circ \phi_i(\mathbb{R}^\star).$$

In this sense, if $|\Phi| = n$ we have an n-ary representation for positive real numbers (though not necessarily the standard n-ary representation, of course).

Example 2.2 The set $\mathbf{Dec} = \{\phi_j(x) = \frac{(10-j)x+(9-j)}{jx+(j+1)} \mid 0 \leq j \leq 9\}$, is a decimal digit set. To see this, note that $\phi_j = \gamma^{-1}\psi_j\gamma$ for $\gamma(x) = \frac{1}{x+1}, \psi_j = \frac{x+j}{10}$; and thatthe set **Dec'** $= \{\psi_j \mid 0 \leq j \leq 9\}$ is the standard decimal representation for the unit interval $[0, 1]$. Note also that γ is a bijection between $\mathbb{R}^\star$ and $[0, 1]$. Now using the fact that the numbers in $[0, 1]$ have a representation in terms of ψ_js, it is easy to check that **Dec** is indeed a digit set.

In this paper in Section 5 we present a binary representation and in section 6 we present a ternary representation.

We will quantify the property of being refining [6, 11, 12, 14]. Since we are dealing with extended set of real numbers, we will consider the image of $\mathbb{R}^\star$ under the one-point compactification of the entire real line. Consider the Möbius map $\mathbf{S}_0(x) = \frac{x-1}{x+1}$. $\mathbf{S}_0$ is a bijection between $\mathbb{R}^\star$ and $[-1, 1]$, with inverse $\mathbf{S}_0^{-1}(x) = \frac{x+1}{-x+1}$. We consider the metric $\rho(x, y) = |\mathbf{S}_0(x) - \mathbf{S}_0(y)|$ on positive real numbers. This metric induces a topology on $\mathbb{R}^\star$, which restricts to the standard topology on $\mathbb{R}^+$.

Because a Möbius map A takes a closed interval $[x, y]$ to either $[A(x), A(y)]$ or $[A(y), A(x)]$, it is natural to define the *diameter* of $[x, y]$ after A by

$$\delta(A, [x, y]) = \rho(A(x), A(y)).$$

Note that, if one fixes A, the resulting function $\delta(A, -)$ is strictly increasing, i.e., if $[x, y] \subsetneqq [x', y']$, then $\delta(A, [x, y]) < \delta(A, [x', y'])$. Also note that:

$$\delta(A \circ B, [x, y]) = \delta(A, [B(x), B(y)]) \tag{2.2.1}$$

(up to reordering of the interval, if B is decreasing). Moreover in terms of $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ it is easy to verify the following equation [12]:

$$\delta(A, [x, y]) = \rho(x, y).|\det A|.\frac{(x+1)(y+1)}{\big((a+c)x + b + d\big)\big((a+c)y + b + d\big)}. \tag{2.2.2}$$

Next we define a measure of contractivity of a set of refining Möbius maps. Let Φ be a set of refining Möbius maps. Let for every natural number k:

$$\mathcal{B}(\Phi, k) = \max\{\delta(\phi_0 \circ \phi_1 \circ \ldots \circ \phi_{k-1}, \mathbb{R}^\star) \mid \phi_0, \ldots, \phi_{k-1} \in \Phi^{<\omega}\}.$$

With this notation we can state and prove the following theorem:

Theorem 2.3 *A finite set Φ of refining increasing Möbius maps is a digit set if both following conditions hold:*

i) $\lim_{j \to \infty} \mathcal{B}(\Phi, j) = 0$,

ii) $\bigcup_{\phi_i \in \Phi} \phi_i(\mathbb{R}^\star) = \mathbb{R}^\star$.

Proof. Proof. Let $\phi_0 \phi_1 \cdots$ be a stream over Φ. We know that $\bigcap_{i=0}^{\infty} \phi_0 \circ \ldots \circ \phi_i(\mathbb{R}^\star)$ is a non-empty closed interval. By (i), the length of the interval is 0, and hence the intersection is a singleton, as desired.

Let $x \in \mathbb{R}^\star$ be given. By (ii) there exists a ϕ_k such that $x \in \phi_k(\mathbb{R}^\star)$. Let $x_0 = \phi_k^{-1}(x)$ and $\delta_0 = \phi_k$. We continue in this way and at each step we find $x_i \in \mathbb{R}^\star$ and $\delta_i \in \Phi$ such that $\delta_i(x_i) = x_{i-1}$. It is easy to see that $x \in \bigcap_{i=0}^{\infty} \delta_0 \circ \ldots \circ \delta_i(\mathbb{R}^\star)$, and hence x is represented by the stream $\delta_0 \delta_1 \cdots$. □

3 Admissible Representations

Theorem 2.3 gives us a criterion for determining whether a given set of Möbius maps is a digit set. Having a digit set not only enables us to represent real numbers by sequences of elements of a finite set, but also gives rise to a class of computable functions with respect to that representation [27]. In this section we introduce the extra restrictions on a digit set in order to make it eligible for computability of the homographic and quadratic functions using algorithms similar to those in [25, 20, 18].

First we introduce some notation. Let Φ be a digit set. For $\alpha \in \Phi^\omega$ let $\alpha\restriction_n$ denote the finite sequence of length n consisting of the first n elements of α. We define the *stream metric* on Φ^ω by

$$\mathbf{d_S}(\alpha, \beta) \triangleq \frac{1}{2^n}, \quad \text{where } \alpha\restriction_n = \beta\restriction_n \text{ but } \alpha\restriction_{n+1} \neq \beta\restriction_{n+1}.$$

It is easy to check that the topology induced by this metric is the usual topology on Φ^ω, namely, the *initial segment* (or *prefix*) topology for Φ. The standard basis for Φ^ω consists of sets U_σ defined by

$$U_\sigma := \{\alpha \in \Phi^\omega \mid \alpha\restriction_n = \sigma\}, \tag{3.0.1}$$

where $\sigma \in \Phi^{<\omega}$. Explicitly, given a finite sequence σ, the basic open (in fact, *clopen*) set U_σ consists of all those α which have σ as an initial segment.

Next we define the notion of an admissible representation for $\mathbb{R}^\star$, derived from [26, 27, 4].

Definition 3.1 A map $p\colon \Phi^\omega \longrightarrow \mathbb{R}^\star$ is an *admissible representation* of $\mathbb{R}^\star$ if the following hold.

i) p is continuous;

ii) p is surjective;

iii) p is *maximal*, i.e. for every (partial) continuous $r\colon \Phi^\omega \longrightarrow \mathbb{R}^\star$, there is a continuous $f\colon \Phi^\omega \longrightarrow \Phi^\omega$ such that $r = p \circ f$.

Intuitively an admissible representation gives rise to functions which are computable with a special kind of Turing machines[28, 27], namely those with potentially infinite input and output.

Similar to Theorem 2.3 we will state a criterion for when a digit set constitutes an admissible representation. N.B. we restrict our attention here to *increasing* Möbius maps. This restriction simplifies our presentation hereafter, but is not essential.

Definition 3.2 Let Φ be a finite set of refining increasing Möbius maps. We call Φ an *admissible digit set* if both following conditions hold:

a. $\lim\limits_{j\to\infty} \mathcal{B}(\Phi, j) = 0$;

b. $\bigcup\limits_{\phi_i \in \Phi} \phi_i(\mathbb{R}^+) = \mathbb{R}^+$.

In Section 6 we present a ternary admissible digit set. A standard example is the set of the maps

$$\{D_k = \left[\begin{smallmatrix} 1+b-k & 1-b+k \\ 1-b-k & 1+b+k \end{smallmatrix}\right] \mid |k| < b\},$$

which constitute a b-ary admissible digit set. This is the digit set that Edalat and Potts use in their development of exact arithmetic[6]. The above definition is similar to the property of *interval containment* of [3], but is weaker (cf. [3, Appendix]). First we prove that an admissible digit set is indeed a restriction on the notion of digit set:

Theorem 3.3 *Any admissible digit set is a digit set.*

Proof. Proof. Let Φ be an admissible digit set. By definition, Theorem 2.3 (i) holds. Now assuming

$$\bigcup_{\phi_i \in \Phi} \phi_i(\mathbb{R}^+) = \mathbb{R}^+, \tag{3.3.1}$$

we should prove that

$$\bigcup_{\phi_i \in \Phi} \phi_i(\mathbb{R}^\star) = \mathbb{R}^\star.$$

Note that $\phi_i(\mathbb{R}^\star)$ is a closed interval, since ϕ_i is strictly increasing and continuous. Also, we know that $(0, +\infty) \subseteq \bigcup\limits_{\phi_i \in \Phi} \phi_i(\mathbb{R}^\star) \subseteq [0, +\infty]$; but a union of closed sets is a closed set, and so the result follows. □

Next we demonstrate how an admissible digit set gives rise to an admissible representation. Let Φ be an admissible digit set. From Definition 3.2 (a) it follows that for every stream $\phi_0\phi_1\cdots$ of elements of Φ, we have $\bigcap\limits_{i=0}^{\infty} \phi_0 \circ \ldots \circ \phi_i(\mathbb{R}^\star) = \{x\}$ for some $x \in \mathbb{R}^\star$. Hence we can define the function $\mathbf{Rep}_\Phi\colon \Phi^\omega \longrightarrow \mathbb{R}^\star$, such that given $\phi_0\phi_1\cdots \in \Phi^\omega$ we have:

$$\phi_0\phi_1\cdots = \{\mathbf{Rep}_\Phi(\bigcap_{i=0}^{\infty} \phi_0 \circ \ldots \circ \phi_i(\mathbb{R}^\star))\}.$$

$\mathbf{Rep}_\Phi\colon \Phi^\omega \longrightarrow \mathbb{R}^\star$ is surjective, by Theorem 3.3. We state some basic properties of $\mathbf{Rep}_\Phi$ in the following lemma. The proof is straightforward.

Lemma 3.4

i) $\mathbf{Rep}_\Phi(\phi_0\phi_1\cdots) = \phi_0(\mathbf{Rep}_\Phi(\phi_1\phi_2\cdots))$

ii) Let $\sigma = \phi_0\phi_1\cdots\phi_k$ *and* U_σ *be the basic open defined by (3.0.1). Then* $\mathbf{Rep}_\Phi(U_\sigma) = \phi_0 \circ \phi_1 \circ \ldots \circ \phi_k(\mathbb{R}^\star)$.

Note that $\mathbf{Rep}_\Phi$ is a total representation[1]. As the following lemma—based on a more general result from [4]—shows, proving the admissibility for total maps boils down to verifying criteria which are simpler than maximality, which is immediate from [4, Corollary 13].

Lemma 3.5 *A total map* $p\colon \Phi^\omega \longrightarrow \mathbb{R}^\star$ *is admissible if it is continuous and has a surjective open restriction.*

In order to apply this theorem, we will have to find a suitable domain for the open restriction of $\mathbf{Rep}_\Phi$. It should be clear that $\mathbf{Rep}_\Phi$ itself is not open. In fact, typically the image $\mathbf{Rep}_\Phi(U_\sigma)$ of a basic open set U_σ is a closed interval (Lemma 3.4 (ii)). What one wants is to remove the endpoints from the basic open sets. This motivates the following definition.

Definition 3.6 Let Φ be an admissible digit set. We say that a stream $\phi_0\phi_1\cdots$ *trails to one side* if there exists k such that $\mathbf{Rep}_\Phi(\phi_k\phi_{k+1}\cdots) \in \{0, +\infty\}$. Otherwise, we say the stream is *non-trailing*. We denote the set of all non-trailing streams by $\Phi^\omega_{\mathsf{nt}}$. Given $\sigma \in \Phi^{<\omega}$ we define the *non-trailing set specified by* σ to be $V_\sigma := U_\sigma \cap \Phi^\omega_{\mathsf{nt}}$.

In other words, a stream $\phi_0\phi_1\cdots$ is non-trailing iff, for every k, the stream $\phi_k\phi_{k+1}\cdots$ is mapped to $\mathbb{R}^+$ (via $\mathbf{Rep}_\Phi$). We will show that every number $x \in \mathbb{R}^+$ is represented by a non-trailing stream. This will be essential in our proof that admissible digit sets yield admissible representations: the function $\mathbf{Rep}_\Phi$ is open when restricted to the non-trailing streams.

Lemma 3.7 *Let* Φ *be an admissible digit set. For every* $x \in \mathbb{R}^+$*, there is a non-trailing stream* $\alpha = \phi_0\phi_1\cdots$ *in* Φ^ω *such that* $\mathbf{Rep}_\Phi(\alpha) = x$.

Proof. Proof. Let x be given. We will use the fact that Φ^ω is a complete metric space, with the metric $\mathbf{d_S}$ defined previously. First, we define a sequence, $\alpha_0, \alpha_1, \ldots$, where each α_i denotes the sequence[2] $\phi^i_0\phi^i_1\cdots$, satisfying the following conditions.

(1) For all i, $\mathbf{Rep}_\Phi(\alpha_i) = x$.

(2) For all i and for all $k \le i+1$, $\mathbf{Rep}_\Phi(\phi^i_k\phi^i_{k+1}\cdots) \in \mathbb{R}^+$. (The stream α_i does not begin to trail off before position $i+2$.)

(3) For all i and for all $j < i$, we have $\mathbf{d_S}(\alpha^i, \alpha^j) \le \frac{1}{2^j}$. In other words, for all $j < i$ and $k \le j$, we have $\phi^i_k = \phi^j_k$.

We define the sequence $\alpha_0, \alpha_1, \ldots$ recursively as follows. By assumption, $x \in \mathbb{R}^+$ and $\bigcup \phi_i(\mathbb{R}^+) = \mathbb{R}^+$. Hence, there is a $\phi^0_0 \in \Phi$ and $y \in \mathbb{R}^+$ such that $\phi^0_0(y) = x$. Now, $\mathbf{Rep}_\Phi$ is surjective, so pick a stream $\phi^0_1\phi^0_2\cdots$ such that $\mathbf{Rep}_\Phi(\phi^0_1\phi^0_2\cdots) = y$. This gives the first stream $\alpha_0 = \phi^0_0\phi^0_1\cdots$. It is easy to confirm the above requirements for α_0.

Suppose that $\alpha_0, \ldots, \alpha_n$ satisfy (1)–(3). Define α_{n+1} as follows. First, for $i \le n$, let $\phi^{n+1}_i = \phi^n_i$. Now, we know that $\mathbf{Rep}_\Phi(\phi^n_{n+1}\phi^n_{n+2}\cdots)$ is in $\mathbb{R}^+$, and so there is a $\phi^{n+1}_{n+1} \in \Phi$ and $y \in \mathbb{R}^+$ such that $\phi^{n+1}_{n+1}(y) = \mathbf{Rep}_\Phi(\phi^n_{n+1}\phi^n_{n+2}\cdots)$. As before, choose a stream $\phi^{n+1}_{n+2}\phi^{n+1}_{n+3}\cdots$ such that $\mathbf{Rep}_\Phi(\phi^{n+1}_{n+2}\phi^{n+1}_{n+3}\cdots) = y$. Again, confirmation of (1)–(3) for α_{n+1} is straightforward.

Condition (3) ensures that our sequence $\alpha_0, \alpha_1, \ldots$ is Cauchy and hence converges, namely to a sequence $\alpha = \phi_0\phi_1\cdots$ such that, for all i and all $j \le i$, $\phi_j = \phi^i_j$. It is easy to see that

[1] This is possible because the co-domain $\mathbb{R}^\star$ is compact [27].

[2] The superscript i in ϕ^i_j is notational. It does not indicate repetition or exponentiation.

$\mathbf{Rep}_\Phi(\alpha) = x$. Proving that α is also non-trailing takes a bit more work, but is not difficult. □

Note that for a finite sequence $\sigma \in \Phi^{<\omega}$, the set V_σ has a semantic nature with respect to the interpretation of the digits in $\mathbb{R}^+$. This is in contrast with the syntactically specified set U_σ. Also the association between the open real intervals and the non-trailing sets specified by finite sequences is implicit in the Lemma 3.7. The following lemma (cf. Lemma 3.4. ii) makes this explicit.

Lemma 3.8

i) If $\alpha \in \Phi^\omega_{nt}$ *and* $\phi_i \in \Phi$ *then* $\phi_i\alpha \in \Phi^\omega_{nt}$.

ii) Let $\sigma = \phi_0\phi_1\cdots\phi_k \in \Phi^{<\omega}$ *and* $A = \phi_0 \circ \phi_1 \circ \ldots \circ \phi_k$. *Then* $\mathbf{Rep}_\Phi(V_\sigma) = (A(0), A(+\infty))$.

Proof.

i) Suppose $\mathbf{Rep}_\Phi(\phi_i\alpha) = 0$. This means (Lemma 3.4. i)

$$\phi_i(\mathbf{Rep}_\Phi(\alpha)) = 0. \tag{3.8.1}$$

Since α is non-trailing, $\mathbf{Rep}_\Phi(\alpha) > 0$. But note ϕ_i is increasing and refining. Hence $\phi_i(\mathbf{Rep}_\Phi(\alpha)) > \phi_i(0) \geq 0$. This is in contradiction with (3.8.1). Consequently $\mathbf{Rep}_\Phi(\phi_i\alpha) \neq 0$. Similarly the assumption $\mathbf{Rep}_\Phi(\phi_i\alpha) = +\infty$ leads to contradiction. This together with the fact that α is non-trailing, entails that $\phi_i\alpha$ should also be non-trailing.

ii) Since $V_\sigma \subset U_\sigma$, by using Lemma 3.4. ii we know that

$$\mathbf{Rep}_\Phi(V_\sigma) \subset \phi_0 \circ \phi_1 \circ \ldots \circ \phi_k(\mathbb{R}^\star). \tag{3.8.2}$$

Let $x \in \mathbf{Rep}_\Phi(V_\sigma)$. This means

$$x = \mathbf{Rep}_\Phi(\sigma\alpha) = \sigma(\mathbf{Rep}_\Phi(\alpha)) \tag{3.8.3}$$

for some $\alpha \in \Phi^\omega_{\mathsf{nt}}$.
Suppose $x = \phi_0 \circ \phi_1 \circ \ldots \circ \phi_k(0) = \sigma(0)$. From this and (3.8.3) we get

$$\sigma(0) = \sigma(\mathbf{Rep}_\Phi(\alpha))$$

Note that σ is a composition of monotone maps and hence it is monotone. Therefore $\mathbf{Rep}_\Phi(\alpha) = 0$, which is in contradiction with the fact that $\alpha \in \Phi^\omega_{\mathsf{nt}}$. Similarly the assumption that $x = \phi_0 \circ \phi_1 \circ \ldots \circ \phi_k(+\infty)$ leads to contradiction and therefore by (3.8.2) we have proven that $x \in \phi_0 \circ \phi_1 \circ \ldots \circ \phi_k(\mathbb{R}^+)$.
To see that $\phi_0 \circ \phi_1 \circ \ldots \circ \phi_k(\mathbb{R}^+) \subset \mathbf{Rep}_\Phi(V_\sigma)$, let $x \in \phi_0 \circ \phi_1 \circ \ldots \circ \phi_k(\mathbb{R}^+)$. This means that there exists $y \in \mathbb{R}^+$ such that $x = \phi_0 \circ \phi_1 \circ \ldots \circ \phi_k(y)$. By Lemma 3.7 we know that there is a non-trailing stream α such that $y = \mathbf{Rep}_\Phi(\alpha)$. Thus $x = \sigma\mathbf{Rep}_\Phi(\alpha)$. Since α is non-trailing by part i it follows that $\sigma\alpha$ is non-trailing. Clearly $\sigma\alpha \in U_\sigma$, and consequently $x = \mathbf{Rep}_\Phi(\sigma\alpha) \in \mathbf{Rep}_\Phi(V_\sigma)$.

□

We are ready to state and prove the main result of this section:

Theorem 3.9 *Let* Φ *be an admissible digit set. Then* $\mathbf{Rep}_\Phi \colon \Phi^\omega \longrightarrow \mathbb{R}^\star$ *is an admissible representation.*

Proof. Proof. We will apply the criteria in Lemma 3.5. By definition, $\mathbf{Rep}_\Phi$ is a total map. In order to prove that $\mathbf{Rep}_\Phi$ is continuous at point γ, assume $\varepsilon > 0$ is given. According to Definition 3.2 (a), there exists an N such that $\mathcal{B}(\Phi, N) < \varepsilon$. Let $\gamma = \phi_0\phi_1 \cdots \phi_{N-1}\gamma'$ and $A = \phi_0 \circ \phi_1 \circ \ldots \circ \phi_{N-1}$. It follows that $\delta(A, \mathbb{R}^\star) \leq \mathcal{B}(\Phi, N) < \varepsilon$. We will show that, for all α such that $\mathbf{d_S}(\alpha, \gamma) < \frac{1}{2^N}$, we have $\rho(\mathbf{Rep}_\Phi(\alpha), \mathbf{Rep}_\Phi(\gamma)) < \varepsilon$.

Accordingly, let such α be given, say, $\alpha = \phi_0\phi_1 \cdots \phi_{N-1}\alpha'$. Then

$$\begin{aligned}\rho(\mathbf{Rep}_\Phi(\alpha), \mathbf{Rep}_\Phi(\gamma)) &= \rho(A(\mathbf{Rep}_\Phi(\alpha')), A(\mathbf{Rep}_\Phi(\gamma')))\\ &= \delta(A, [\mathbf{Rep}_\Phi(\alpha'), \mathbf{Rep}_\Phi(\gamma')])\\ &\leq \delta(A, \mathbb{R}^\star) < \varepsilon.\end{aligned}$$

This completes the proof that $\mathbf{Rep}_\Phi$ is continuous.

We claim that the restriction of $\mathbf{Rep}_\Phi$ to the set

$$G := \Phi^\omega_{\mathsf{nt}} \cup \mathbf{Rep}_\Phi^{-1}(0) \cup \mathbf{Rep}_\Phi^{-1}(+\infty).$$

is an open surjection onto $\mathbb{R}^\star$. We denote this restriction by $f :\subseteq \Phi^\omega \longrightarrow \mathbb{R}^\star$. Clearly, f is a surjection (Lemma 3.7).

Finally we show that f is an open map. Let $W \overset{\text{open}}{\subseteq} G$ be a basic open set in the subspace topology. Thus there exists $U_\sigma \overset{\text{open}}{\subseteq} \Phi^\omega$ such that

$$W = U_\sigma \cap G = (U_\sigma \cap \Phi^\omega_{\mathsf{nt}}) \cup \underbrace{(U_\sigma \cap \mathbf{Rep}_\Phi^{-1}(0))}_{W_1} \cup \underbrace{(U_\sigma \cap \mathbf{Rep}_\Phi^{-1}(+\infty))}_{W_2}$$

Note that $U_\sigma \cap \Phi^\omega_{\mathsf{nt}} = V_\sigma$ and hence we write $W = V_\sigma \cup W_1 \cup W_2$.

Assume $y \in f(W)$. Then $y = \mathbf{Rep}_\Phi(\beta)$ for some $\beta \in W$. If $\beta \in V_\sigma$, then y is in the interior of an open interval in $\mathbb{R}^\star$. This is because by Lemma 3.8. ii $f(V_\sigma)$ is an open interval in $\mathbb{R}^+$. Therefore in this case we can find an open subinterval of $f(W)$ containing y.

Next suppose $\beta \in W_1$. Since $\beta \in U\sigma$ we can write $\beta = \sigma\beta'$ for some $\beta' \in \Phi^\omega$. Since $\beta \in W_1$ we deduce $y = \mathbf{Rep}_\Phi(\beta) = 0$ and consequently

$$\sigma\mathbf{Rep}_\Phi(\beta') = \mathbf{Rep}_\Phi(\beta) = 0.$$

Since σ — now considered as a Möbius map — is increasing and refining, we should have $\mathbf{Rep}_\Phi(\beta') = 0$ and therefore $\sigma(0) = 0$. As a consequence of this and by Lemma 3.8. ii, we get $f(V_\sigma) = (0, \sigma(+\infty))$. Thus

$$y \in [0, \sigma(+\infty)) \subset \{0\} \cup f(V_\sigma) \subset f(W), \quad [0, \sigma(+\infty)) \overset{\text{open}}{\subseteq} \mathbb{R}^\star.$$

If $\beta \in W_2$ one can similarly show that $y \in (\sigma(0), +\infty] \overset{\text{open}}{\subseteq} f(W)$.

Thus F maps basic open sets to open intervals and hence it is open. □

The above proof for Theorem 3.9 is based on totality of $\mathbf{Rep}_\Phi$. According to [27, Theorem 4.1.15], no representation of the (non-compactified) real numbers can be total. Thus the above proof can not be used to show the admissibility of the restriction of $\mathbf{Rep}_\Phi$ as a representation of $\mathbb{R}^+$.

4 Algebraic structure on Φ^ω

By results of [26], if Φ is an admissible digit set then any continuous function on $\mathbb{R}^\star$ can be lifted to a continuous function on Φ^ω. This means that in particular for addition and multiplication we can write continuous functions on Φ^ω that computes them. This general result, while useful, does not suffice for doing actual formal verifications in, say, Coq. For that, one needs an explicit presentation of the so-called homographic and quadratic algorithms (c.f. [25]). Here, we present the homographic algorithm here and confirm that it is productive.

In this section we present the homographic algorithm for an admissible representation. We assume we are given an admissible digit set Φ. By *homographic algorithm*[3] we mean a function $H\colon \mathbb{M} \times \Phi^\omega \longrightarrow \Phi^\omega$ such that, for all $\alpha \in \Phi^\omega$ and refining Möbius maps A, we have

$$\mathbf{Rep}_\Phi(H(A,\alpha)) = A(\mathbf{Rep}_\Phi(\alpha)) \tag{4.0.1}$$

For $\phi \in \Phi$ and $A \in \mathbb{M}$, we introduce the following shorthand:

$$A \sqsubseteq \phi := A(\mathbb{R}^+) \subseteq \phi(\mathbb{R}^+)$$

We further fix an ordering on the finite set Φ and denote its elements by $\phi_0, \phi_1, \ldots, \phi_{l-1}$. A finite sequence of digits, then, will be denoted $\phi_{i_0}\phi_{i_1}\cdots\phi_{i_n}$, and similarly for streams.

We aim to define our function $H\colon \mathbb{M} \times \Phi^\omega \longrightarrow \Phi^\omega$ so that it satisfies the following.

$$H(A, \phi_i\alpha) := \begin{cases} \phi_0\ H(\phi_0^{-1} \circ A, \phi_i\alpha) & \textbf{if } A \sqsubseteq \phi_0 \\ \phi_1\ H(\phi_1^{-1} \circ A, \phi_i\alpha) & \textbf{else if } A \sqsubseteq \phi_1 \\ \quad\vdots & \\ \phi_{l-1}\ H(\phi_{l-1}^{-1} \circ A, \phi_i\alpha) & \textbf{else if } A \sqsubseteq \phi_{l-1} \\ H(A \circ \phi_i, \alpha) & \textbf{otherwise}. \end{cases} \tag{4.0.2}$$

Each of the first k branches of the homographic algorithm is called an *emission step*, while the last branch is called an *absorption step*. Note that the inverse Möbius maps ϕ_j^{-1} are not necessarily total functions. Nonetheless, $A \sqsubseteq \phi_j$ implies that $A(\mathbb{R}^+)$ is a subset of the domain of ϕ_j^{-1}, and so $\phi_j^{-1} \circ A$ is well defined and refining in each clause in which it appears. Furthermore, since A and ϕ_j are both refining, so is $A \circ \phi_j$.

In order to define H as above, we first define a family of sequences representing partial computations of H. Explicitly, for each $A \in \mathbb{M}$ and stream $\alpha = \phi_{i_0}\phi_{i_1}\phi_{i_2}\ldots$, we define a function $h^{A,\alpha}\colon \mathbb{N} \longrightarrow \mathbb{M} \times \Phi^{<\omega} \times \mathbb{N}$, where the first projection (denoted $\mathsf{M}^{A,\alpha}$) represents the Möbius map to be used in the next step of computation, the second projection (denoted $\mathsf{em}^{A,\alpha}$) represents the digits emitted so far and the third projection (denoted $\mathsf{ab}^{A,\alpha}$) notes how much of the input has been absorbed so far. For readability, we omit the superscripts for M, em and ab below.

$$h^{A,\alpha}(0) = \langle A,\, [],\, 0\rangle$$

$$h^{A,\alpha}(n+1) = \begin{cases} \langle \phi_0^{-1} \circ \mathsf{M}(n),\, \mathsf{em}(n)\phi_0,\, \mathsf{ab}(n)\rangle & \textbf{if } \mathsf{M}(n) \sqsubseteq \phi_0 \\ \langle \phi_1^{-1} \circ \mathsf{M}(n),\, \mathsf{em}(n)\phi_1,\, \mathsf{ab}(n)\rangle & \textbf{else if } \mathsf{M}(n) \sqsubseteq \phi_1 \\ \quad\vdots & \\ \langle \phi_{l-1}^{-1} \circ \mathsf{M}(n),\, \mathsf{em}(n)\phi_{l-1},\, \mathsf{ab}(n)\rangle & \textbf{else if } \mathsf{M}(n) \sqsubseteq \phi_{l-1} \\ \langle \mathsf{M}(n) \circ \phi_{i_{\mathsf{ab}(n)}},\, \mathsf{em}(n),\, \mathsf{ab}(n)+1\rangle & \textbf{otherwise} \end{cases}$$

Again, we call the first k cases *emission steps* and the last an *absorption step*. In each case, we alter the Möbius map for the next step of the computation, either by post-composing with an appropriate ϕ_i^{-1} (in emission steps) or pre-composing with the next digit of α (in absorption steps). In emission steps, we append the appropriate ϕ_i to the output so far. In the absorption step, the output is unchanged, but we note that we have absorbed another digit of the input by incrementing $\mathsf{ab}^{A,\alpha}(n)$.

[3]We use the term "homographic", because the original algorithm given by Gosper [8] for computing addition and multiplication of two continued fractions was called homographic algorithm. What Gosper considered a homographic function, we call a refining Möbius map.

Let $\Phi^{\leq\omega}$ be the union of the set of finite sequences $\Phi^{<\omega}$ with the streams Φ^{ω} and let $\lesssim$ denote the initial segment ordering on $\Phi^{\leq\omega}$ (so $\alpha \lesssim \beta$ iff α is an initial segment of β). Clearly, for each A, α and n, we have

$$\mathsf{em}^{A,\alpha}(n) \lesssim \mathsf{em}^{A,\alpha}(n+1).$$

Since $\Phi^{\leq\omega}$ is a complete partial order with respect to $\lesssim$, we may take the directed join $\bigsqcup_{n=0}^{\omega} \mathsf{em}^{A,\alpha}(n)$. We wish to define

$$H(A,\alpha) = \bigsqcup_{n=0}^{\omega} \mathsf{em}^{A,\alpha}(n),$$

but we must show that the directed join is in Φ^{ω} (i.e., is an infinite sequence).

The join $\bigsqcup \mathsf{em}^{A,\alpha}(n)$ satisfies the following: $\mathsf{length}(\bigsqcup \mathsf{em}^{A,\alpha}(n)) > j$ iff there is an n such that $\mathsf{length}(\mathsf{em}^{A,\alpha}(n)) > j$ and, furthermore,

$$\Big(\bigsqcup_{n=0}^{\omega} \mathsf{em}^{A,\alpha}(n)\Big)_j = \big(\mathsf{em}^{A,\alpha}(n)\big)_j. \tag{4.0.3}$$

Hence, to show that the join is an infinite sequence, we must show that, for every j, there is an n such that $\mathsf{length}(\mathsf{em}^{A,\alpha}(n)) > j$.

In order to show that this is, indeed, the case, we introduce a quantity which characterizes the redundancy of the admissible digit set. The idea is to view the overlapping of the range of the digits as a measure for redundancy of the representation[4].

Definition 4.1 Let Φ be an admissible digit set. We define the *redundancy* of Φ as

$$\mathsf{red}(\Phi) = \min\{\rho(\phi_i(0), \phi_j(+\infty)) \mid \phi_i, \phi_j \in \Phi,\ \phi_i(0) \neq \phi_j(+\infty)\}. \tag{4.1.1}$$

The redundancy has the following important property, essential for showing that our proposed definition for $H(A,\alpha)$ indeed yields streams over Φ. With this lemma in hand, we can show that absorption steps will only be iterated a finite number of times, followed by an emission.

Lemma 4.2 *Let $A \in \mathbb{M}$ such that $\delta(A, \mathbb{R}^\star) < \mathsf{red}(\Phi)$. Then there exists $0 \leq i < k$ such that $A \sqsubseteq \phi_i$.*

Proof. Proof. By (3.2.b), we know that $A(\mathbb{R}^+) \subseteq \bigcup_{i=0}^{k-1} \phi_i(\mathbb{R}^+)$. Hence, either there is an i such that $\phi_i(0) \in A(\mathbb{R}^+)$ or there is an i such that $A(\mathbb{R}^+) \subseteq \phi_i(\mathbb{R}^+)$. In the latter case, we have $A \sqsubseteq \phi_i$.

For the former case, suppose there is an $x \in A(\mathbb{R}^+)$ such that $x = \phi_i(0)$ for some $\phi_i \in \Phi$. By assumption, $\delta(A, \mathbb{R}^\star) < \mathsf{red}(\Phi)$ and so $\mathbf{S}_0 \circ A(+\infty) - \mathbf{S}_0 \circ A(0) < \mathsf{red}(\Phi)$, where $\mathbf{S}_0 \colon \mathbb{R}^\star \longrightarrow [-1,1]$ is the bijection from Section 2. It follows that

$$\mathbf{S}_0 \circ A(+\infty) - \mathbf{S}_0(x) < \mathsf{red}(\Phi),$$
$$\mathbf{S}_0(x) - \mathbf{S}_0 \circ A(0) < \mathsf{red}(\Phi),$$

so $\mathbf{S}_0 \circ A(\mathbb{R}^+) \subsetneq [\mathbf{S}_0(x) - \mathsf{red}(\Phi), \mathbf{S}_0(x) + \mathsf{red}(\Phi)]$. Since $x \in \mathbb{R}^+$, there is a $\phi_j \in \Phi$ with $x \in \phi_j(\mathbb{R}^+)$. But minimality of $\mathsf{red}(\Phi)$ in (4.1.1) means that the end points $\mathbf{S}_0 \circ \phi_j(0)$ and $\mathbf{S}_0 \circ \phi_j(+\infty)$ are at least at a distance $\mathsf{red}(\Phi)$ from $\mathbf{S}_0(x)$. In other words,

$$[\mathbf{S}_0(x) - \mathsf{red}(\Phi), \mathbf{S}_0(x) + \mathsf{red}(\Phi)] \subseteq [\mathbf{S}_0 \circ \phi_j(0), \mathbf{S}_0 \circ \phi_j(+\infty)],$$

and so $A \sqsubseteq \phi_j$. □

We now complete the argument that our definition of H indeed yields a function $\mathbb{M} \times \Phi^{\omega} \longrightarrow \Phi^{\omega}$.

[4]In fact, as defined, the redundancy may be less than the diameter of the minimal overlapping ranges of the digits, but the definition as given is simple and suffices.

Theorem 4.3 *Let $A \in \mathbb{M}$ and $\alpha \in \Phi^\omega$ be given and let $\beta = \bigsqcup \mathsf{em}^{A,\alpha}(n)$. Then $\beta \in \Phi^\omega$.*

Proof. Proof. We prove by induction that, for every j, there exists an n such that $\mathsf{length}(\mathsf{em}^{A,\alpha}(n)) \geq j$. The base case ($j = 0$) is trivial, so we suppose that the claim is true for some j and prove it for $j+1$.

Let n be given, then, such that $\mathsf{length}(\mathsf{em}^{A,\alpha}(n)) \geq j$ and let B be the matrix of coefficients for $\mathsf{M}^{A,\alpha}(n)$. Let

$$B = \left[\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right], M = \left[\begin{smallmatrix} 1 & 1 \\ b_{11}+b_{21} & b_{12}+b_{22} \end{smallmatrix}\right].$$

We consider two cases, with respect to the sign of $\det M$:

$\det M \geq 0$: In this case M denotes an increasing Möbius map and as a consequence

$$M(0) \geq M(+\infty) = \frac{1}{b_{11}+b_{21}}. \tag{4.3.1}$$

Since $\lim_{j\to\infty} \mathcal{B}(\Phi, j) = 0$, there exists N such that

$$\mathcal{B}(\Phi, N) < \frac{\mathsf{red}(\Phi)\ (b_{11}+b_{21})^2}{|\det B|}. \tag{4.3.2}$$

Take $J = n + N + 1$. We claim that $\mathsf{length}(\mathsf{em}^{A,\alpha}(J)) \geq j+1$. Let $\alpha = \phi_{i_0}\phi_{i_1}\phi_{i_2}\ldots$ and let

$$C = \phi_{i_{\mathsf{ab}^{A,\alpha}(n)}} \circ \phi_{i_{\mathsf{ab}^{A,\alpha}(n)+1}} \circ \ldots \circ \phi_{i_J}.$$

The Möbius map C, then, is constructed by taking the composition of the next N digits of the input stream α.

We may assume that every step from n to $n+N$ (inclusive) is an absorption step, so that

$$h^{A,\alpha}(J-1) = \langle B \circ C,\ \mathsf{em}^{A,\alpha}(n),\ \mathsf{ab}^{A,\alpha}(n) + N\rangle.$$

Now, by our choice of N, we have

$$\delta(C, \mathbb{R}^+) < \frac{\mathsf{red}(\Phi)\ (b_{11}+b_{21})^2}{|\det B|}.$$

We calculate

$$\begin{aligned} \delta(B \circ C, \mathbb{R}^+) &= \delta(B, C(\mathbb{R}^+)) \\ &= \delta(C, \mathbb{R}^+).|\det B|.M(0).M(+\infty) && \text{by (2.2.2)} \\ &\leq \delta(C, \mathbb{R}^+).\frac{|\det B|}{(b_{11}+b_{21})^2} && \text{by (4.3.1)} \\ &< \mathsf{red}(\Phi). \end{aligned}$$

Hence we can apply Lemma 4.2 and obtain ϕ_i such that $B \circ C \sqsubseteq \phi_i$. Thus, we see that the Jth step is an emission step, as desired.

$\det M < 0$: In this case M denotes a decreasing Möbius map and

$$M(+\infty) < M(0) = \frac{1}{b_{12}+b_{22}}.$$

Therefore taking N such that

$$\mathcal{B}(\Phi, N) < \frac{\mathsf{red}(\Phi)\ (b_{12}+b_{22})^2}{|\det B|},$$

we can continue the reasoning as in the previous case.

□

We have thus proved that H is a function of the right type, but it remains to be seen that H satisfies the equation (4.0.2). This is the next task at hand.

Lemma 4.4 *Let $H(A, \alpha) = \bigsqcup h^{A,\alpha}(n)$. Then H satisfies (4.0.2).*

Proof. Proof. Let A and α be given, and let $h^{A,\alpha}(1) = \langle B, \overline{\phi}, k\rangle$ (here, $\overline{\phi}$ is either an empty sequence or a singleton and k either 0 or 1). One can show that, for every n,

$$h^{A,\alpha}(n+1) = \langle \mathsf{M}^{B,\beta}(n), \overline{\phi}\mathsf{em}^{B,\beta}(n), k + \mathsf{ab}^{B,\beta}(n)\rangle.$$

The proof proceeds by induction on n, and is perfectly straightforward, so we omit it here.

Now, suppose that $H(A, \alpha)$ is an emission step for digit ϕ_i. According to (4.0.2), we should show that

$$\begin{aligned}\big(H(A, \alpha)\big)_0 &= \phi_i \\ \big(H(A, \alpha)\big)_{j+1} &= \big(H(\phi_i^{-1} \circ A, \alpha)\big)_j\end{aligned}$$

The former is easy: Since, by assumption, i is the least number such that $A \sqsubseteq \phi_i$, we have

$$h^{A,\alpha}(1) = \langle \phi_i^{-1} \circ A, \phi_i, 0\rangle.$$

Apply equation 4.0.3.

For the latter, let j be given. Also, let $B = \phi_i^{-1} \circ A$. By definition of H, there is an n such that

$$\begin{aligned}\big(H(A, \alpha)\big)_{j+1} &= \big(\mathsf{em}^{A,\alpha}(n+1)\big)_{j+1} \\ &= \big(\phi_i \mathsf{em}^{B,\alpha}(n)\big)_{j+1} \\ &= \big(\mathsf{em}^{B,\alpha}(n)\big)_j \\ &= \big(H(B, \alpha)\big)_j.\end{aligned}$$

The proof for the case that $H(A, \alpha)$ is an absorbing step proceeds similarly. We omit it here. □

We have two tasks remaining, then. First, we wish to show that our function H is productive (when we fix the Möbius map A). Second, we must show that H actually does what it is supposed to, namely, that it computes the Möbius map A.

We adapt the definition of productivity found in [23] to our setting, Φ^ω. Productivity is the condition that finite portions of the output depend only on finite portions of the input, so that the function does not look arbitrarily deep into the input stream to compute initial segments of the output.

Definition 4.5 A (total) function $f \colon \Phi^\omega \longrightarrow \Phi^\omega$ is *productive*, if

$$\forall \alpha \in \Phi^\omega \; \forall j \in \mathbb{N} \; \exists k \in \mathbb{N} \; \forall \beta \; \big(\beta{\restriction_k} = \alpha{\restriction_k} \implies f(\beta){\restriction_j} = f(\alpha){\restriction_j}\big). \tag{4.5.1}$$

Intuitively, f is productive if for any $k \in \mathbb{N}$ the first k elements of its output are produced in a finite amount of time. More explicitly, if f is productive, then the first k positions of the output depend only on a finite initial segment of the input. Clearly, productivity is just the same as continuity with respect to the metric $\mathbf{d_S}$.

Theorem 4.6 *Let A be a non-singular Möbius map. The function $H(A, -)$ is productive.*

Proof. Proof. Let $\alpha \in \Phi^\omega$ and $j \in \mathbb{N}$ be given, and we must show that there is a k such that, for all β satisfying $\beta\restriction_k = \alpha\restriction_k$, we have $H(A,\beta)\restriction_j = H(A,\alpha)\restriction_j$.

Pick n such that $\mathsf{length}(\mathsf{em}^{A,\alpha}(n)) \geq j$ and let $k = \mathsf{ab}^{A,\alpha}(n)$, the number of digits of input of α absorbed by the nth step of the computation of $H(A,\alpha)$. Let β be given such that $\beta\restriction_k = \alpha\restriction_k$. We claim that, for every $m \leq n$,

$$h^{A,\beta}(m) = h^{A,\alpha}(m). \tag{4.6.1}$$

This will suffice to complete the proof, since $H(A,\beta)\restriction_j = \mathsf{em}^{A,\beta}(n)\restriction_j$.

We prove (4.6.1) by induction on m, with the case $m = 0$ trivial. The inductive step for $h^{A,\alpha}(m+1)$ an emission step is also easy. If $h^{A,\alpha}(m+1)$ is an absorption step, then we use the fact that $\mathsf{ab}^{A,\alpha}(m) \leq k$ to conclude that $(\alpha)_{\mathsf{ab}^{A,\alpha}(m)} = (\beta)_{\mathsf{ab}^{A,\beta}(m)}$, and so the result follows. □

We now proceed to the proof that H is correct, i.e., that for all A and α, we have

$$A(\mathbf{Rep}_\Phi(\alpha)) = \mathbf{Rep}_\Phi(H(A,\alpha)).$$

The right hand side is the unique element of the intersection of all the $\phi_{j_0} \circ \ldots \circ \phi_{j_n}(\mathbb{R}^\star)$, where $\phi_{j_0} \cdots \phi_{j_n}$ is an initial segment of the output. The following lemma is essential in proving that the left hand side is an element of that intersection.

Lemma 4.7 *Let A and $\alpha = \phi_{i_0}\phi_{i_1}\cdots$ be given and let $n \in \mathbb{N}$. Let $h^{A,\alpha}(n) = \langle B_n, \overline{\phi}_n, k_n\rangle$, where $\overline{\phi}_n = \phi_{j_0} \cdots \phi_{j_{m_n}}$. Then for all $\beta \in \Phi^\omega$,*

$$A(\mathbf{Rep}_\Phi(\alpha\restriction_{k_n}\ \beta)) \in \phi_{j_0} \circ \ldots \circ \phi_{j_{m_n}}(\mathbb{R}^\star).$$

Proof. Proof. We proceed by induction on n, with the base case trivial. Suppose, then, that the claim holds for n and that the $n+1$st step emits $\phi_{j_{m_{n+1}}}$. Then, it must be the case that $B_n \sqsubseteq \phi_{j_{m_{n+1}}}$, i.e.,

$$B_n(\mathbb{R}^+) \subseteq \phi_{j_{m_{n+1}}}(\mathbb{R}^+).$$

Hence, for every $\beta \in \Phi^\omega$, we have $B_n(\mathbf{Rep}_\Phi(\beta)) \in \phi_{j_{m_{n+1}}}(\mathbb{R}^\star)$.

It is easy to show by induction that

$$B_n = \phi^{-1}_{j_{m_n}} \circ \ldots \circ \phi^{-1}_{j_0} \circ A \circ \phi_{i_0} \circ \ldots \circ \phi_{i_{k_n-1}}.$$

Hence

$$\begin{aligned} A(\mathbf{Rep}_\Phi(\alpha\restriction_{k_n}\ \beta)) &= A \circ \phi_{i_0} \circ \ldots \circ \phi_{i_{k_n-1}}(\mathbf{Rep}_\Phi(\beta)) \\ &= \phi_{j_0} \circ \ldots \circ \phi_{j_{m_n}} \circ B_n(\mathbf{Rep}_\Phi(\beta)) \\ &\in \phi_{j_0} \circ \ldots \circ \phi_{j_{m_{n+1}}}(\mathbb{R}^\star). \end{aligned}$$

This completes the proof of the inductive step for emissions. Suppose, then, that the $n+1$st step is instead an absorption step. We must show that, for all β,

$$A(\mathbf{Rep}_\Phi(\phi_{i_0}\cdots\phi_{i_{k_n}}\beta)) \in \phi_{j_0} \circ \ldots \circ \phi_{j_{m_n}}(\mathbb{R}^\star).$$

But, by inductive hypothesis, for all γ,

$$A(\mathbf{Rep}_\Phi(\phi_{i_0}\cdots\phi_{i_{k_n}}\gamma)) \in \phi_{j_0} \circ \ldots \circ \phi_{j_{m_n}}(\mathbb{R}^\star).$$

Apply this to $\gamma = \phi_{i_{k_n}}\beta$. □

Theorem 4.8 *For every $A \in \mathbb{M}$, the function $H(A,-)\colon \Phi^\omega \longrightarrow \Phi^\omega$ computes A, in the sense that*

$$A(\mathbf{Rep}_\Phi(\alpha)) = \mathbf{Rep}_\Phi(H(A,\alpha)).$$

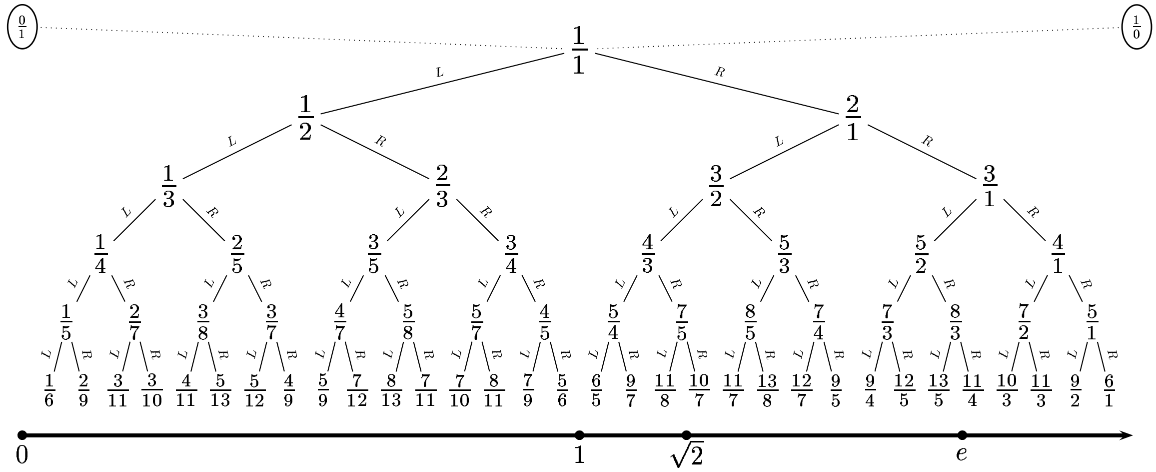

Figure I: The Stern–Brocot tree

Proof. Proof. Let $\alpha = \phi_{i_0}\phi_{i_1}\cdots$ and $\mathbf{Rep}_\Phi(H(A,\alpha)) = \phi_{j_0}\phi_{j_1}\cdots$. We must show that, for every k,

$$A(\mathbf{Rep}_\Phi(\alpha)) \in \phi_{j_0} \circ \ldots \circ \phi_{j_k}(\mathbb{R}^\star).$$

Let k be given and pick n such that $\mathsf{length}(\mathsf{em}^{A,\alpha}(n)) = j_k + 1$, so that $\phi_{j_0} \circ \ldots \circ \phi_{j_k} = \mathsf{em}^{A,\alpha}(n)$. Apply Lemma 4.7. □

Theorem 4.8 shows that the homographic algorithm can be used to evaluate Möbius maps applied to a stream of digits. Potts [20] and Edalat and Potts [6] show how one can generalize the structure of the algorithm for computing on *expression trees*. An expression tree corresponds to a real function. The simplest expression tree corresponds to the quadratic algorithm. More complex expression trees correspond to transcendental real functions [6]. The approach we took in order to prove the productivity and correctness of homographic algorithm, can be generalized to prove the productivity and correctness of some simple expression trees corresponding such as the quadratic algorithm. The situation for more complex expression trees has yet to be investigated.

5 Stern–Brocot Representation

Stern–Brocot tree (Figure I) presents an elegant way of encoding positive rational numbers as elements of the set $\mathbf{SB} = \{\mathbf{L}, \mathbf{R}\}^{<\omega}$ [10, 1, 2, 18]. That the streams of L's and R's yield Cauchy sequences of real numbers is a well-known part of the Stern–Brocot folklore. Here, we will show that this representation is a digit set, in the sense of Section 2.

The Stern–Brocot encoding explicitly involves the Möbius maps

$$\phi_{\mathbf{L}} = \frac{x}{x+1},$$
$$\phi_{\mathbf{R}} = x + 1,$$

which suggests a representation of the real numbers via the set $\Phi_{\mathbf{2}} = \{\phi_{\mathbf{L}}, \phi_{\mathbf{R}}\}$. In keeping with our previous development, we will show that $\Phi_{\mathbf{2}}$ is a digit set (though not admissible), so that $\mathbb{R}^\star$ can be represented by streams over $\Phi_{\mathbf{2}}$. In keeping with our previous development, then, we throw away the finite sequences and consider only the set $\Phi_{\mathbf{2}}^\omega$ of streams over $\Phi_{\mathbf{2}}$.

Clearly both $\phi_{\mathbf{L}}$ and $\phi_{\mathbf{R}}$ are refining and increasing Möbius maps. Moreover note that $\phi_{\mathbf{L}}(\mathbb{R}^\star) = [0, 1]$ and $\phi_{\mathbf{R}}(\mathbb{R}^\star) = [1, +\infty]$, so $\phi_{\mathbf{L}}(\mathbb{R}^\star) \cup \phi_{\mathbf{R}}(\mathbb{R}^\star) = \mathbb{R}^\star$, as condition (ii) in Theorem 2.3 requires. We must show, then, that $\lim_{k\to\infty} \mathcal{B}(\Phi_{\mathbf{2}}, k) = 0$. We will sketch the proof of this fact.

First, one observes the diameter of $\mathbb{R}^\star$ under applications of constant sequences. Explicitly, one can show that, for all $k > 0$,

$$\delta(\phi_{\mathbf{L}}^k, \mathbb{R}^\star) = \delta(\phi_{\mathbf{R}}^k, \mathbb{R}^\star) = \frac{2}{k+1} \tag{5.0.1}$$

Moreover, it is easy to prove by induction the following two facts for all k and compositions $A = \phi_1 \circ \ldots \circ \phi_k$.

(i) $A(0) = 0$ iff $A = \phi_{\mathbf{L}} \circ \phi_{\mathbf{L}} \circ \ldots \circ \phi_{\mathbf{L}}$, and otherwise, $A(0) \geq \frac{1}{k}$.

(ii) $A(+\infty) = +\infty$ iff $A = \phi_{\mathbf{R}} \circ \phi_{\mathbf{R}} \circ \ldots \circ \phi_{\mathbf{R}}$, and otherwise, $A(+\infty) \leq k$.

With these facts in hand, we can give the upper bound for contractivity of the digits of $\Phi_{\mathbf{2}}$ at each step.

Lemma 5.1 *For every $k > 0$, $\mathcal{B}(\Phi_{\mathbf{2}}, k) = \frac{2}{k+1}$.*

Proof. Proof. We will prove the lemma by induction on k, with the base case trivial.

Suppose that the claim holds for k and let $\phi_0\phi_1 \cdots \phi_k$ be given. Let $A = \phi_1 \circ \ldots \circ \phi_k$ and $B = \phi_0 \circ A$. We must show that $\delta(B, \mathbb{R}^\star) \leq \frac{2}{k+2}$, given that $\delta(A, \mathbb{R}^\star) \leq \frac{2}{k+1}$.

We consider the case where $\phi_0 = \mathbf{L}$. We may assume that some $\phi_i \neq \phi_{\mathbf{L}}$, since otherwise the result holds by (5.0.1). Let $\psi_{\mathbf{L}} = \frac{x+1}{2x+1}$, a decreasing Möbius map.

$$\begin{aligned}
\delta(B, \mathbb{R}^\star) &= \delta(\phi_{\mathbf{L}}, [A(0), A(+\infty)]) && \text{by (2.2.1)}\\
&= \rho(A(0), A(+\infty)) \frac{(A(0)+1)(A(+\infty)+1)}{(2A(0)+1)(2A(+\infty)+1)} && \text{by (2.2.2)}\\
&= \rho(A(0), A(+\infty)) \psi_{\mathbf{L}}(A(0)) \psi_{\mathbf{L}}(A(+\infty))\\
&\leq \rho(A(0), A(+\infty)) \big(\psi_{\mathbf{L}}(A(0))\big)^2 && (\star)\\
&\leq \rho(A(0), A(+\infty)) \left(\psi_{\mathbf{L}}\left(\frac{1}{k}\right)\right)^2 && \text{by (i)}\\
&= \delta(A, \mathbb{R}^\star) \cdot \frac{k+1}{k+2}\\
&\leq \frac{2}{k+1} \cdot \frac{k+1}{k+2} && \text{by (IH)}\\
&= \frac{2}{k+2}.
\end{aligned}$$

In the calculation above, the inequality $(\star)$ comes from the fact that $\psi_{\mathbf{L}}$ is decreasing and A increasing.

The case in which $\phi_0 = \phi_{\mathbf{R}}$ is strictly analogous, using $\psi_{\mathbf{R}} = \frac{x+1}{x+2}$. □

The following corollary is immediate.

Corollary 5.2 *The set $\Phi_{\mathbf{2}} = \{\phi_{\mathbf{L}}, \phi_{\mathbf{R}}\}$ is a digit set.*

Hence, one may represent the interval $\mathbb{R}^\star$ via the digit set $\Phi_{\mathbf{2}}$. Naturally the next step would be to apply the algorithms given in [18] to compute arithmetic operations for this representation. But this is not possible. For example if we want to add $1 - \frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$ using the quadratic algorithm of [18], after absorbing any initial segment of the input sequences we are still unable to output an element of the output because we don't know whether the result is bigger or smaller than 1. With the terminology from the Section 3, we can prove the following lemma which implies that the binary Stern–Brocot representation is not sufficient for computations.

Lemma 5.3

i) Multiplication by 2 *is not continuous on* $\Phi_{\mathbf{2}}^{\omega}$*.*

ii) $\mathbf{Rep}_{\Phi_{\mathbf{2}}}$ *is not an admissible representation.*

Proof. Proof.

i) Suppose there is a continuous map $\Gamma\colon \Phi_{\mathbf{2}}^{\omega} \longrightarrow \Phi_{\mathbf{2}}^{\omega}$ that *"computes the multiplication by* 2*"*, that is to say:

$$\forall \alpha \in \Phi_{\mathbf{2}}^{\omega}.\ \mathbf{Rep}_{\Phi_{\mathbf{2}}}(\Gamma\alpha) = 2 \cdot \mathbf{Rep}_{\Phi_{\mathbf{2}}}(\alpha). \tag{5.3.1}$$

Because Γ is continuous at $\beta = \phi_{\mathbf{L}}\phi_{\mathbf{L}}\phi_{\mathbf{R}}^{\infty}$, (which interprets to $\frac{1}{2}$), given $N > 0$, there is $m \geq 0$ such that

$$\forall \alpha \in \Phi_{\mathbf{2}}^{\omega}.\ \mathbf{d_S}(\alpha, \beta) < \frac{1}{2^m} \Rightarrow \mathbf{d_S}(\Gamma\alpha, \Gamma\beta) < \frac{1}{2^N} \tag{5.3.2}$$

Now, either $\Gamma\beta = \phi_{\mathbf{R}}\phi_{\mathbf{L}}^{\infty}$ or $\Gamma\beta = \phi_{\mathbf{L}}\phi_{\mathbf{R}}^{\infty}$, since these are the only sequences which interpret to 1. Suppose $\Gamma\beta = \phi_{\mathbf{R}}\phi_{\mathbf{L}}^{\infty}$ (the other case is similar). Then choose $\alpha = \phi_{\mathbf{L}}^2\phi_{\mathbf{R}}^{m+1}\phi_{\mathbf{L}}^{\infty}$. Clearly $\mathbf{d_S}(\alpha, \beta) < \frac{1}{2^m}$. By an easy induction on m one can obtain $\mathbf{Rep}_{\Phi_{\mathbf{2}}}(\alpha) = \frac{m+1}{2m+3}$. Consequently by (5.3.1) we have $\Gamma\alpha \in \mathbf{Rep}_{\Phi_{\mathbf{2}}}^{-1}(\frac{2m+2}{2m+3})$. Therefore the first digit of α must be $\phi_{\mathbf{L}}$, and this means $\mathbf{d_S}(\Gamma\alpha, \Gamma\beta) = 1$ which is in contradiction with (5.3.2).

ii) Suppose $\mathbf{Rep}_{\Phi_{\mathbf{2}}}$ is admissible. Since $f(x) = 2x$ is a continuous map on $\mathbb{R}^{\star}$, then the map $f \circ \mathbf{Rep}_{\Phi_{\mathbf{2}}} \colon \Phi_{\mathbf{2}}^{\omega} \longrightarrow \mathbb{R}^{\star}$ is continuous. Hence by the maximality property of the admissible representation there should exists a continuous map $\Gamma\colon \Phi_{\mathbf{2}}^{\omega} \longrightarrow \Phi_{\mathbf{2}}^{\omega}$ s.t. $\mathbf{Rep}_{\Phi_{\mathbf{2}}} \circ \Gamma = f \circ \mathbf{Rep}_{\Phi_{\mathbf{2}}} = 2 \cdot \mathbf{Rep}_{\Phi_{\mathbf{2}}}$. This is in contradiction with part i.

□

6 Admissible Stern–Brocot Representation

In this section, we extend the digit set $\Phi_{\mathbf{2}}$ of the previous section, to get an admissible digit set. We will use the theory developed in Section 3 and add one digit to the set $\Phi_{\mathbf{2}}$ of previous section.

Lemma 5.3 shows that the set $\mathbf{SB}^{\omega}$ does not contain enough redundancy for computability purposes. Looking back at the Definition 3.2, we see that the source of the difficulty lies in the fact that $\phi_{\mathbf{R}}(\mathbb{R}^+) \cup \phi_{\mathbf{L}}(\mathbb{R}^+) \neq \mathbb{R}^+$. We will "patch" this by adding an extra map. Consider the map

$$\phi_{\mathbf{M}}(x) := \frac{2x+1}{x+2}.$$

In this section we shall prove that the set $\Phi_{\mathbf{3}} = \{\phi_{\mathbf{L}}, \phi_{\mathbf{R}}, \phi_{\mathbf{M}}\}$ is an admissible digit set. Note that $1 \in \phi_{\mathbf{M}}(\mathbb{R}^+)$. Thus the condition (b) of Definition 3.2 holds for $\Phi_{\mathbf{3}}$. We must show that $\lim_{j\to\infty} \mathcal{B}(\Phi_{\mathbf{3}}, j) = 0$.

The proof of this fact is very similar to what we did in Section 5. We already have a bound for $\delta(A, \mathbb{R}^{\star})$ for those Möbius maps A made up by compositions of $\phi_{\mathbf{R}}$ and $\phi_{\mathbf{L}}$. We will show that bound applies to all compositions of $\Phi_{\mathbf{3}}$.

In fact, one can show that, if $A = \phi_1 \circ \ldots \circ \phi_k$ is a composition involving ϕ_M, then

$$\frac{1}{k+1} \leq A(0) \text{ and } A(+\infty) \leq k+1.$$

We use that fact to show the following.

Lemma 6.1 *Let* $A = \phi_0 \circ \ldots \circ \phi_k$*, where some* $\phi_i = \phi_{\mathbf{M}}$*. Then* $\delta(A, \mathbb{R}^{\star}) \leq \frac{2}{k+2}$*.*

The proof follows same reasoning as in Lemma 5.1, with an extra case for $\phi_0 = \phi_{\mathbf{M}}$, where $\psi_{\mathbf{M}}$ is the constant map $\frac{1}{3}$.

Since we already knew that compositions $\phi_0 \circ \ldots \circ \phi_k$ not involving $\phi_{\mathbf{M}}$ have diameter bounded by $\frac{2}{k+1}$, we have the following.

Lemma 6.2 *For every $k > 0$, $\mathcal{B}(\Phi_{\mathbf{3}}, k) \le \frac{2}{k+1}$.*

Hence the following corollary:

Corollary 6.3 *$\Phi_{\mathbf{3}}$ is an admissible digit set. Consequently, $\mathbf{Rep}_{\Phi_{\mathbf{3}}}$ is an admissible representation.*

We can then apply the results of Section 4. For each refining Möbius map A, there is a productive function $\overline{\mathbf{SB}}^{\omega} \longrightarrow \overline{\mathbf{SB}}^{\omega}$ computing A. In this particular case, the condition $A \sqsubseteq \psi$ will boil down to comparing two line segments $ax + b$ and $cx + d$ for positive values of x. According to the result of comparison we may output a digit or ask for more input [18].

7 Conclusion

In this paper we have quantified the property of redundancy for a representation of real numbers and have applied this redundancy in order to obtain a generic proof of productivity of the exact arithmetic algorithms. We have shown the proof in detail in the case of homographic algorithm. The method applied is generalizable to proving the correctness of larger classes of exact arithmetic functions namely quadratic algorithm. It remains to be seen how far this method is applicable for the general normalization algorithm of Potts[20] for expression trees.

In the second part of the paper we have presented a representation for positive real numbers which is a modification of the binary Stern–Brocot representation for rational numbers. There are some novelties in this new representation. First, this representation is given by three Möbius maps, two of which are parabolic LFT's and hence are not considered in the framework of Potts and Edalat[20, 6]. Moreover this representation show why the convergence criterion given in [12, Theorem 3.5] is not a necessary condition, since **con L** = **con R** = 1 (for the definition of **con** see [12]).

It is interesting to study other possible enhancements of the binary stern–Brocot representation. A good candidate will be to instead of the map $\phi_{\mathbf{M}}$ of Section 6, consider the map

$$\phi'_{\mathbf{M}}(x) \; = \begin{cases} \frac{1}{2-x} & x \le 1, \\ \frac{2x-1}{x} & 1 < x. \end{cases}$$

Adding this map is inspired by the relation between Stern–Brocot tree and the greatest common divisor algorithm and corresponds to the "greedy" Euclid's algorithm [13, Exercise 4.5.3.30]. The map $\phi'_{\mathbf{M}}$ is a refining and piecewise Möbius map. It can be shown that a ternary representation using $\phi_{\mathbf{L}}, \phi_{\mathbf{R}}$ and $\phi'_{\mathbf{M}}$ is an example of a non-LFT IFS-representation.

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