The Muddy Children: A logic for public announcement

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February 10, 2007

Outline

The muddy children

Quincy Prescott Hughes

Baba: “At least one of you is muddy.”
Baba: “Are you muddy?”
Quincy: “I don’t know.”
Prescott: “I don’t know.”
Hughes: “I don’t know.”

The muddy children

Quincy Prescott Hughes

Baba: “Are you muddy?”
Quincy: “Aha! What if I wasn’t muddy?”
Quincy: “Then Prescott would not have seen any muddy kids.”
Quincy: “Prescott would have said ’yes’ last time!”
Quincy: “I must be muddy.”

The muddy children


Quincy


Prescott


Hughes

Baba: "Are you muddy?"
Quincy: "Yes."
Prescott: "Yes."
Hughes: "I don't know."

The muddy children


Quincy


Prescott


Hughes

When Baba said, "At least one kid is muddy," every kid knew that. . . but they didn't know that the other kids knew that!

Public announcements of φ tell you φ, everyone knows φ, everyone knows that everyone knows φ, . . .

Modal operators

A modal operator $\Box$ is a logical operator.
We use it to build new formulas from old.

If φ is a formula, then so is $\Box\varphi$.

We use modal operators to express lots of concepts, including:

Necessarily φ.	$\Box\varphi$	$\Diamond\varphi$	Possibly φ.
φ will always be true.	$G\varphi$	$F\varphi$	Eventually φ.
φ is provable.	$\mathrm{Prov}\,\varphi$	??	φ is not refutable.
It ought to be φ	$O\varphi$	$P\varphi$	φ is permitted.
I know φ	$K\varphi$	??	I think φ is possible.

Each operator $\Box$ has a dual, $\Diamond = \neg\Box\neg$.

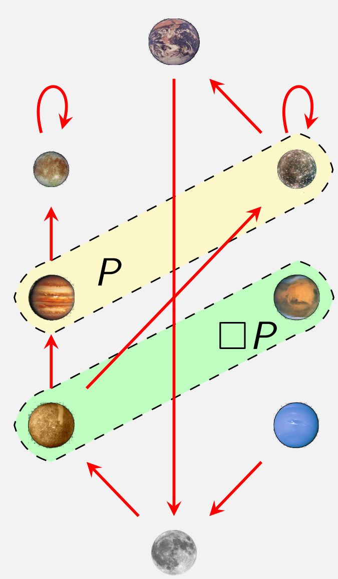
Kripke semantics

Models for modal logics are based on "possible world" semantics.
Let $\mathcal{W}$ be a set of worlds with a graph.

Write $w \models P$ if P is true at world w.

$$
\begin{array}{rll}
w \models \varphi \wedge \psi & \text{iff} & w \models \varphi \text{ and } w \models \psi \\
w \models \varphi \vee \psi & \text{iff} & w \models \varphi \text{ or } w \models \psi \\
w \models \varphi \rightarrow \psi & \text{iff} & w \models \psi \text{ or } w \not\models \varphi \\
w \models \neg\varphi & \text{iff} & w \not\models \varphi
\end{array}
$$

$w \models \Box\varphi$ iff for every $w \longrightarrow w'$, $w' \models \varphi$.



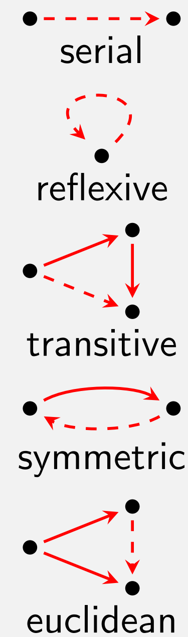
Modal axioms and frame conditions

Axioms on $\Box$ correspond to conditions on the graph.

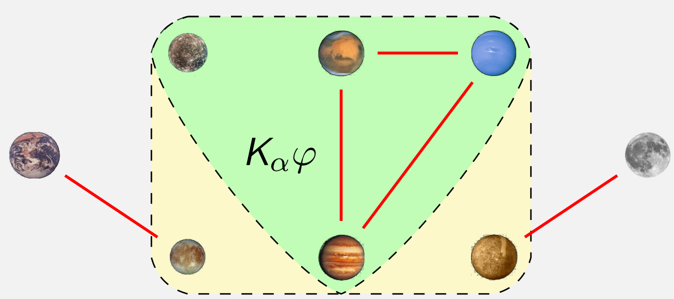
Name	Axiom	Graph is...
(D)	$\Box\varphi \to \Diamond\varphi$	serial
(M)	$\Box\varphi \to \varphi$	reflexive
(4)	$\Box\varphi \to \Box\Box\varphi$	transitive
(B)	$\varphi \to \Box\Diamond\varphi$	symmetric
(5)	$\Diamond\varphi \to \Box\Diamond\varphi$	euclidean

If $\Box$ satisfies (M), (4) and (B), then the graph is an equivalence relation.

- Write $w \overset{\alpha}{\text{——}} w'$.
- Don't bother to draw loops.



The epistemic operator



For each agent α, we introduce an operator K_α.
$K_\alpha\varphi$ <u>means</u> "α knows φ."
Each α has its own graph, too.

An edge $w \overset{\alpha}{\text{——}} w'$ <u>means</u> "α can not distinguish w from w'."

$w \models K_\alpha\varphi$ iff for every $w \overset{\alpha}{\text{——}} w'$, $w' \models \varphi$.

More on K_α


Quincy


Prescott


Hughes

$K_\alpha\varphi$ <u>means</u> "α knows φ."
What does $\neg K_\alpha \neg\varphi$ mean? α considers that φ is possible.
What about $K_\alpha K_\beta \varphi$? α knows that β knows that φ.
For instance, Quincy knows that Hughes knows that Prescott is muddy. In other words, $K_Q K_H(P \text{ is muddy})$.

Properties of K_α


Quincy

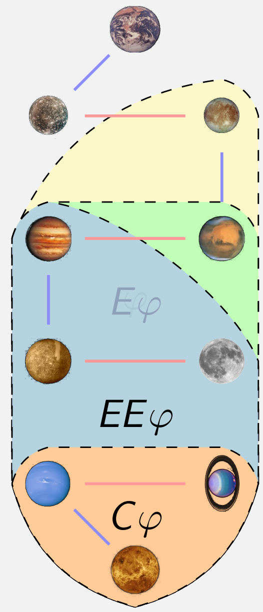

Prescott


Hughes

$K_\alpha\varphi \rightarrow \varphi$	knowledge is true
$K_\alpha\varphi \rightarrow K_\alpha K_\alpha\varphi$	positive introspection
$\neg K_\alpha\varphi \rightarrow K_\alpha\neg K_\alpha\varphi$	negative introspection
$K_\alpha(\varphi \rightarrow \psi) \rightarrow (K_\alpha\varphi \rightarrow K_\alpha\psi)$	distributivity

Universal and common knowledge

- Universal knowledge ($E\varphi$):
  - Everyone knows φ.
  - No one-step paths outside of φ.
- Universal knowledge of universal knowledge ($EE\varphi$):
  - Everyone knows that everyone knows φ.
  - No two-step paths outside of φ.
  - No one-step paths outside of universal knowledge.
- Common knowledge ($C\varphi$):
  - Everyone knows that everyone knows that...that everyone knows φ.
  - No paths out of φ.



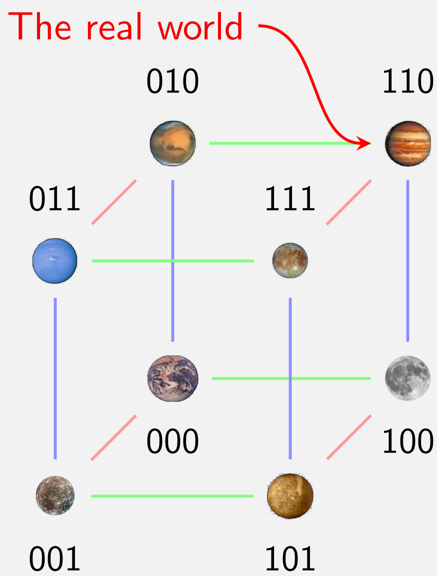
The muddy children
Modal logics
The epistemic operator
A logic for public announcement

Back to the kids

Eight possible worlds.
0 - clean
1 - muddy

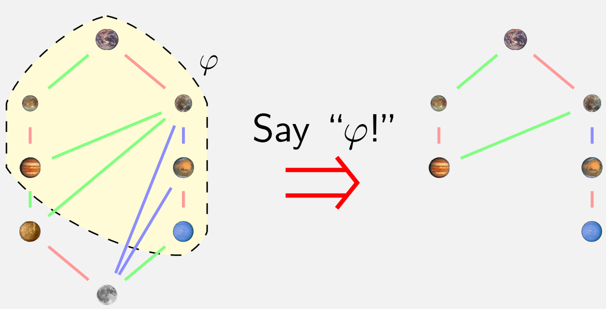
Hughes The Muddy Children:A logic for public announcement

Back to the kids



Quincy cannot distinguish a world where he is muddy from one where he isn't.
World 110 is indistinguishable from 010.
Quincy's epistemic relation.
Prescott's relation.
And Hughes's relation.

Dynamic features



What happens when someone announces φ?

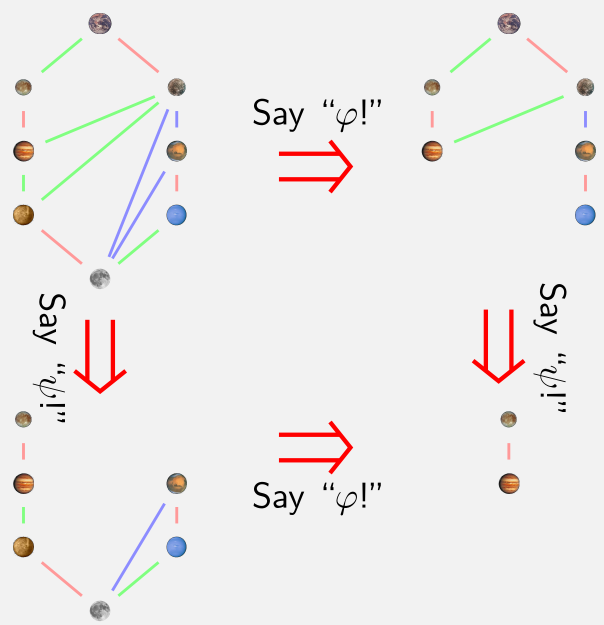
Everyone learns that φ was true when announced.

So the $\neg\varphi$ worlds are unimportant. Take 'em out!

Edges, too!

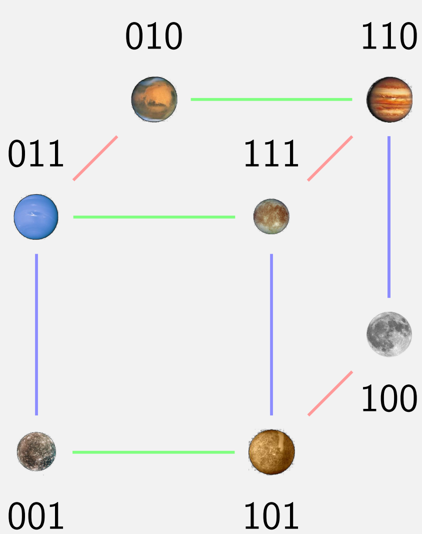
Information reduces uncertainty by eliminating possibilities.

A model of possible models!



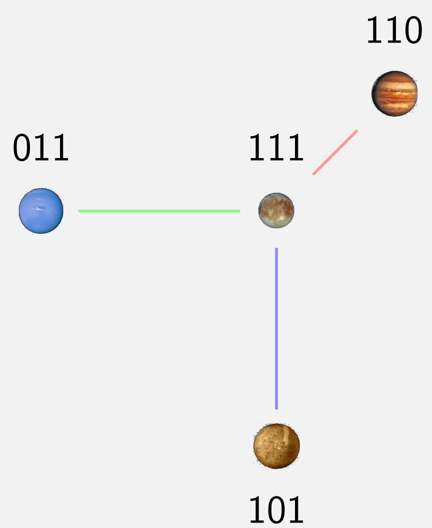
- Announcing φ changes the model.
- Announcing ψ changes it another way.
- Get a transition system on models.
- Another Kripke frame!

Resolving the muddy children



Baba: "At least one of you is muddy."
World 000 is inconsistent with this announcement.
We remove it from the model.
Before $w_{110} \models E\varphi$.
Now $w_{110} \models C\varphi$.

Resolving the muddy children



Baba: "Are you muddy?"
Quincy: "I don't know."
Prescott: "I don't know."
Hughes: "I don't know."

Remove world 100!
Remove world 010!
Remove world 001!

A much simpler model!

Resolving the muddy children

But now:
$w_{110} \models K_Q($"Q is muddy"$)$!
Baba: "Are you muddy?"
Quincy: "Yes!"
Prescott: "Yes!"
Hughes: "I don't know."

Quincy knows Quincy is muddy: remove 011 and 111.

Prescott knows Prescott is muddy: remove 111 and 101.

Resolving the muddy children

110

Baba: "Are you muddy?"
Quincy: "Yes!"
Prescott: "Yes!"
Hughes: "No!"

References

- Stanford Encyclopedia of Philosophy. . http://plato.stanford.edu/entries/logic-modal/
- Benthem, J. v. "Language, logic, and communication". In Logic in Action, J. van Benthem, et al. ILLC, 2001.
- Benthem, J. v. "One is a Lonely Number". http://staff.science.uva.nl/~johan/Muenster.pdf