Simulations in Coalgebra

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University of Nijmegen

Outline

I. Simulations, bisimulations, two-way simulations

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II. Orders on functors

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III. Lax relation lifting

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V. DPCO structure on final coalgebras

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II. Orders on functors

III. Lax relation lifting

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V. DPCO structure on final coalgebras

VI. Summary

Simulations, etc.

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Bisimilarity	$a \underline{\leftrightarrow} b$	$\Leftrightarrow$	$\exists R \,.\, aRb$ and R, R^{op} are simulations
Two-way similarity	$a \eqsim b$	$\Leftrightarrow$	$a \lesssim b$ and $b \lesssim a$

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Consider $FX = 1 + \mathbb{N} \times X$.

Final F-coalgebra: (possibly finite) sequences over $\mathbb{N}$.

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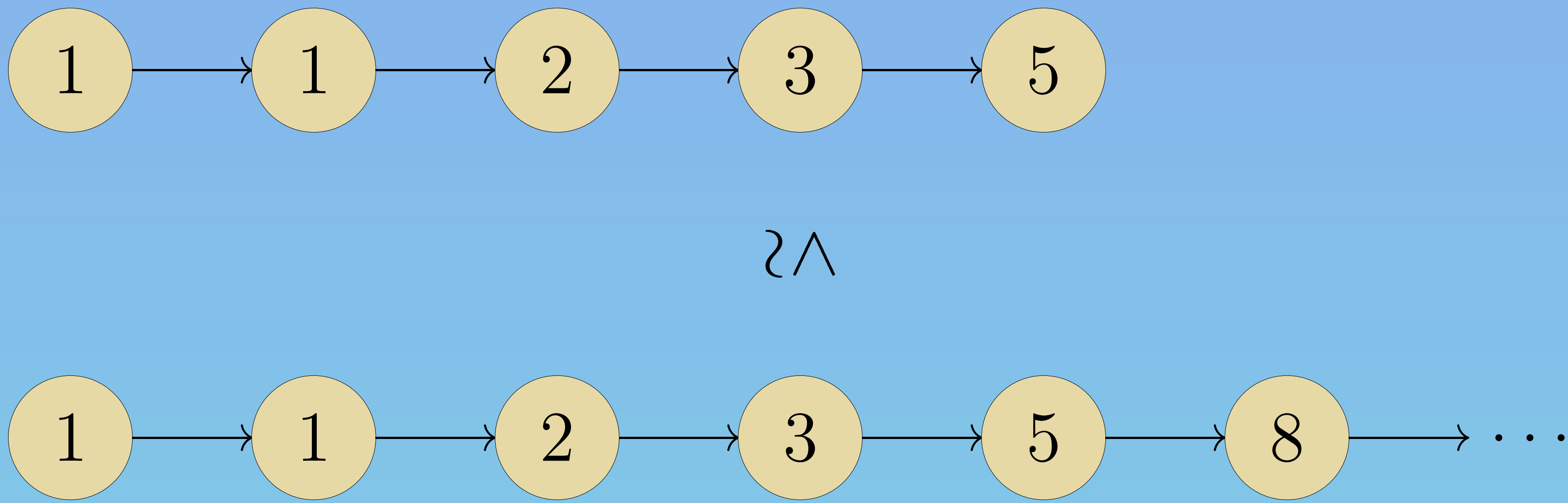
“Standard” similarity

$\sigma \lesssim_1 \tau \Leftrightarrow \sigma$ is a prefix of τ.



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$\sigma \lesssim_2 \tau \Leftrightarrow \mathsf{len}(\sigma) = \mathsf{len}(\tau)$ and for each $n < \mathsf{len}(\sigma)$,
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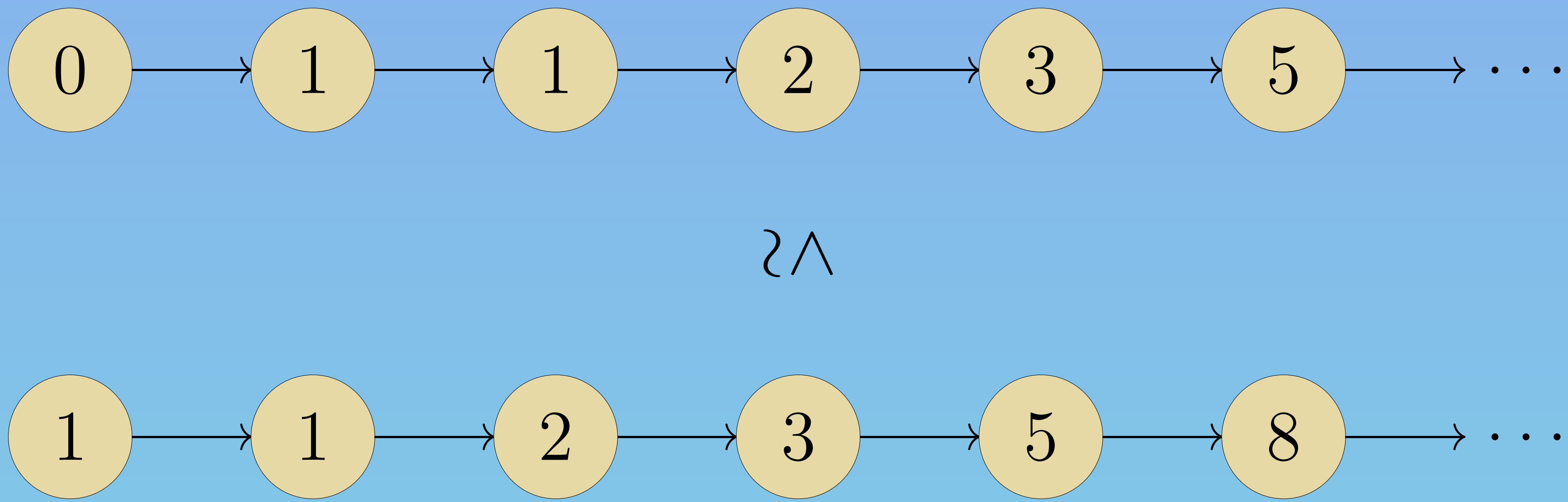

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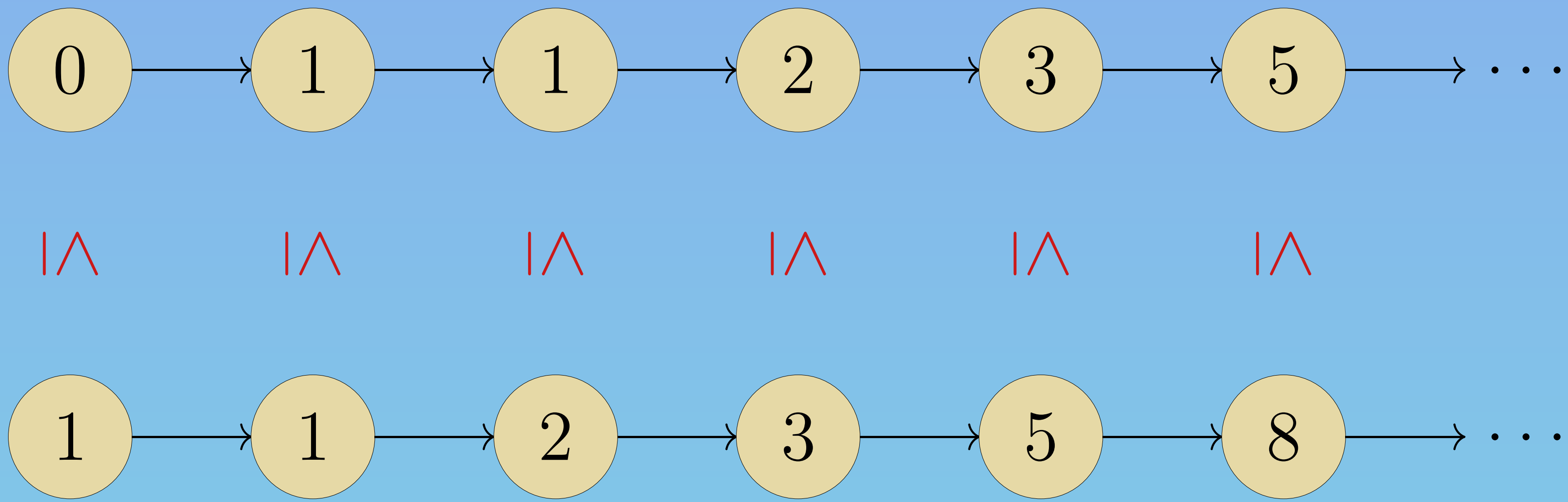

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Similarity via composition

$\sigma(\lesssim_2 \circ \lesssim_1)\tau \Leftrightarrow \mathsf{len}(\sigma) \leq \mathsf{len}(\tau)$ and for all $n \leq \mathsf{len}(\sigma)$,
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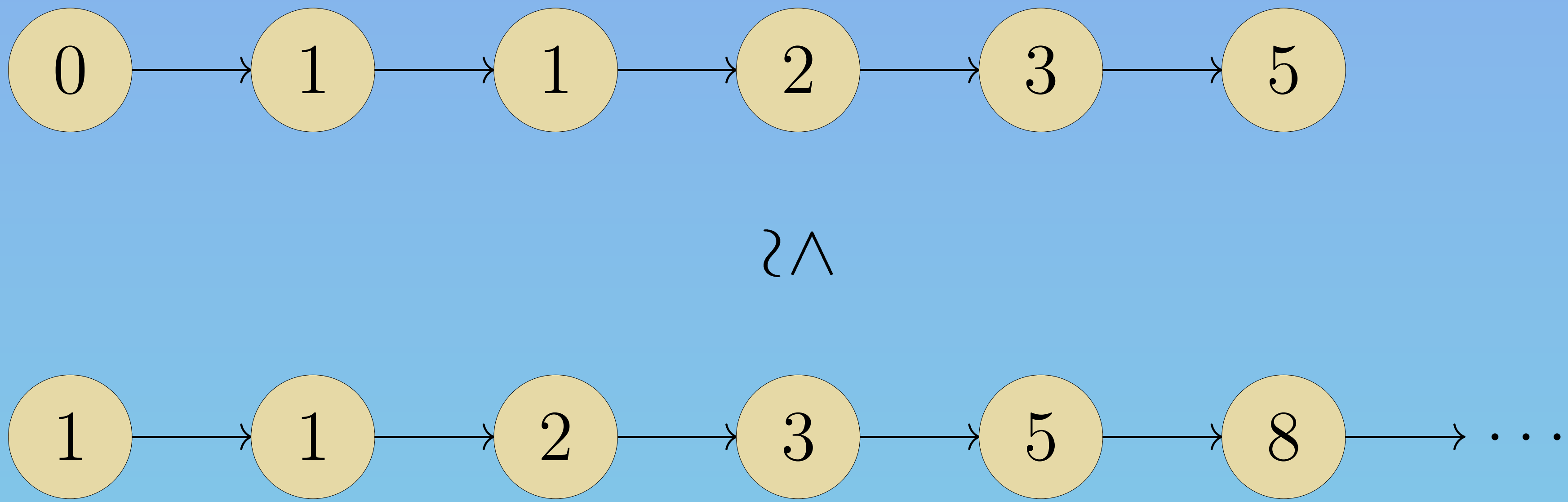


Sequences

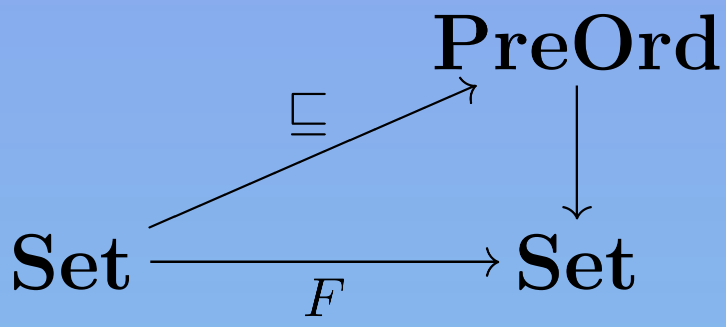
Consider $FX = 1 + \mathbb{N} \times X$.

Final F-coalgebra: (possibly finite) sequences over $\mathbb{N}$.

What structure suffices to describe these examples of similarity?

Our starting point: Orders on functors

An *order* on a functor $F : \mathbf{Set} \to \mathbf{Set}$ is a functor $\sqsubseteq : \mathbf{Set} \to \mathbf{PreOrd}$ such that this diagram commutes.



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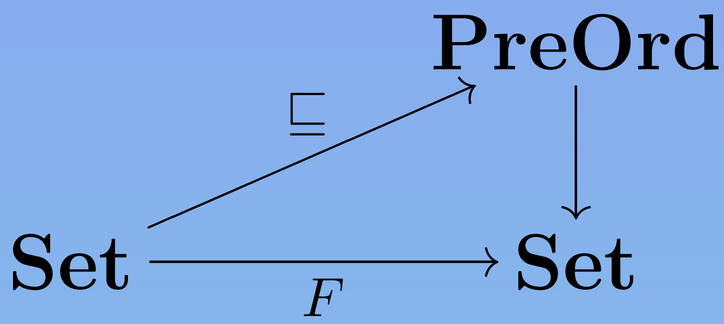

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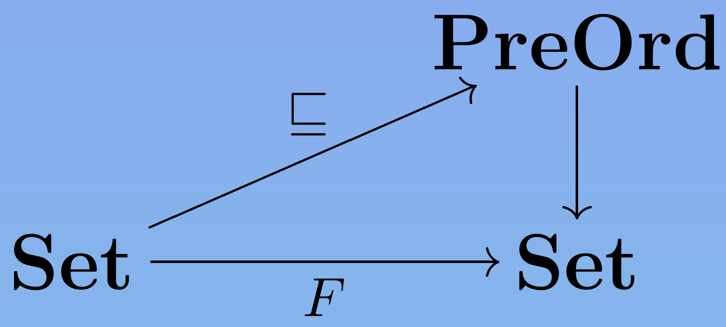

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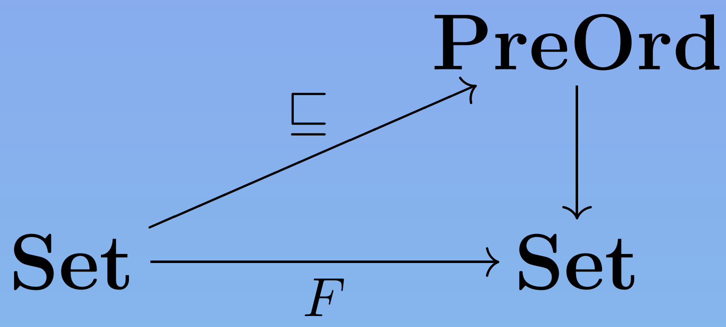

This means:

- For each set X, we have a preorder $\sqsubseteq_X$ on FX;
- For each map $f : X \rightarrow Y$, the map $Ff : FX \rightarrow FY$ is monotone.

An order on F yields a notion of F-similarity.

Excursion: bisimulations

A functor $F : \mathbf{Set} \to \mathbf{Set}$ has a (canonical) associated relation lifting:

$$\begin{array}{ccc}
\mathbf{Rel} & \xrightarrow{\mathbf{Rel}(F)} & \mathbf{Rel} \\
\downarrow & & \downarrow \\
\mathbf{Set} \times \mathbf{Set} & \xrightarrow[F \times F]{} & \mathbf{Set} \times \mathbf{Set}
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This can be defined via image factorization:

$$\begin{array}{ccc}
FR & \twoheadrightarrow & \mathbf{Rel}(F)(R) \\
\downarrow & & \downarrow \\
F(A\times B) & \xrightarrow[\langle F\pi_1, F\pi_2\rangle]{} & FA\times FB
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Excursion: bisimulations

A bisimulation over F-coalgebras $\langle A,\, \alpha \rangle$ and $\langle B,\, \beta \rangle$ is a $\mathbf{Rel}(F)$-coalgebra:

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It is a relation R such that

$$aRb \quad \Rightarrow \quad \alpha(a)\ \mathbf{Rel}(F)(R)\ \beta(b).$$

Lax relation liftings

$$\begin{array}{ccc}
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$$\exists x', y' \; . \; x \sqsubseteq x' \; \mathbf{Rel}(F)(R) \; y' \sqsubseteq y.$$

Simulations

A *simulation* on $\langle A,\ \alpha\rangle$ and $\langle B,\ \beta\rangle$ is a $\mathbf{Rel}_{\sqsubseteq}(F)$-coalgebra over $\alpha \times \beta$.

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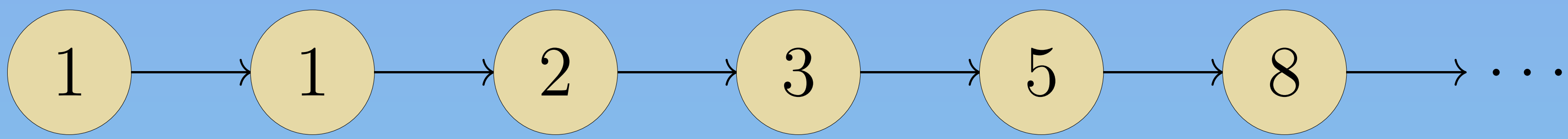
For any pair of coalgebras, the greatest simulation $\lesssim$ exists.

Examples

Consider $FX = 1 + \mathbb{N} \times X$.



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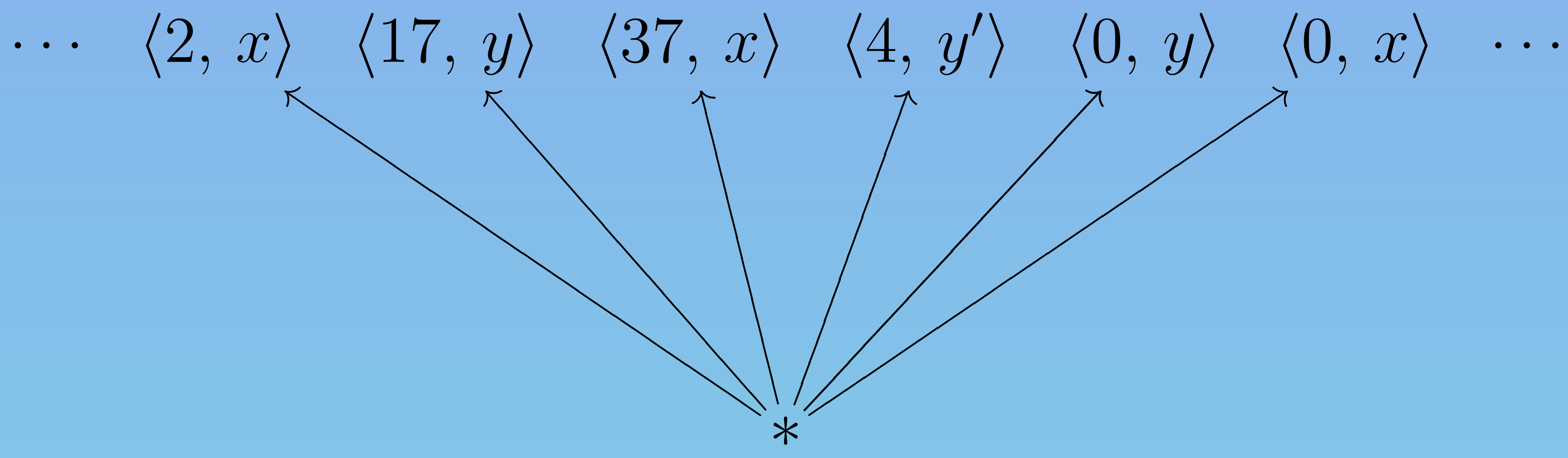

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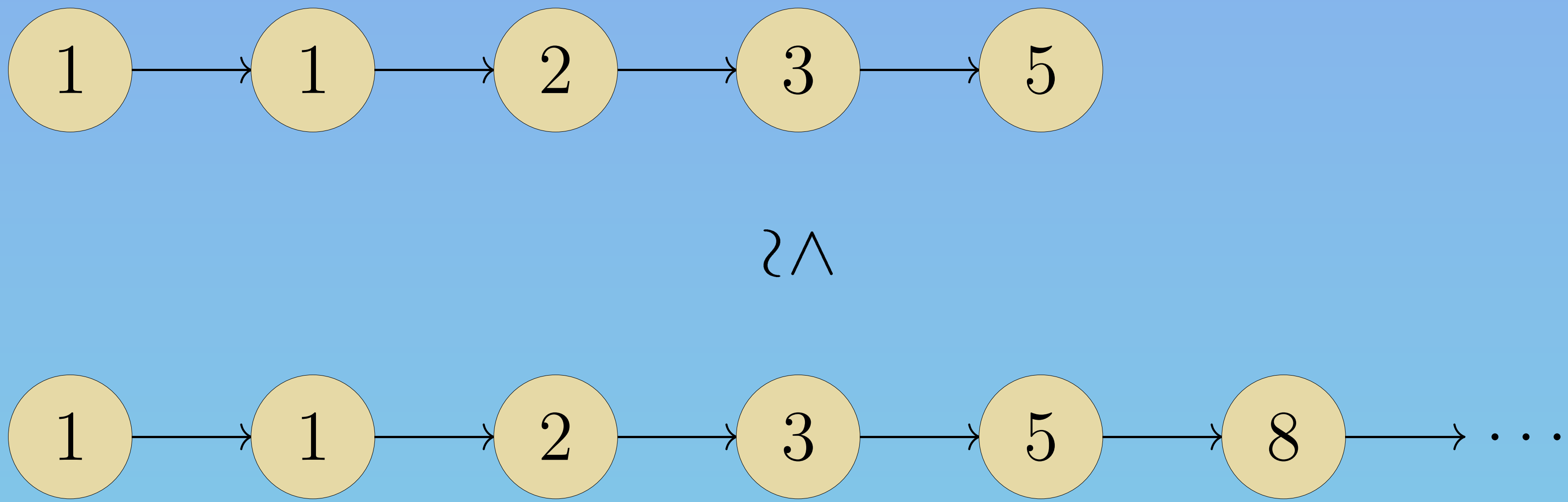

Examples

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Define $x \sqsubseteq_2 y \Leftrightarrow x = y = *$ or $\pi_1(x) \leq \pi_1(y)$.

$$
\begin{array}{cc}
 & \vdots \\
 & \uparrow \\
 & \{5\} \times X \\
 & \uparrow \\
 & \{4\} \times X \\
 & \uparrow \\
 & \{3\} \times X \\
 & \uparrow \\
 & \{2\} \times X \\
 & \uparrow \\
\{*\} & \{1\} \times X \\
 & \uparrow \\
 & \{0\} \times X
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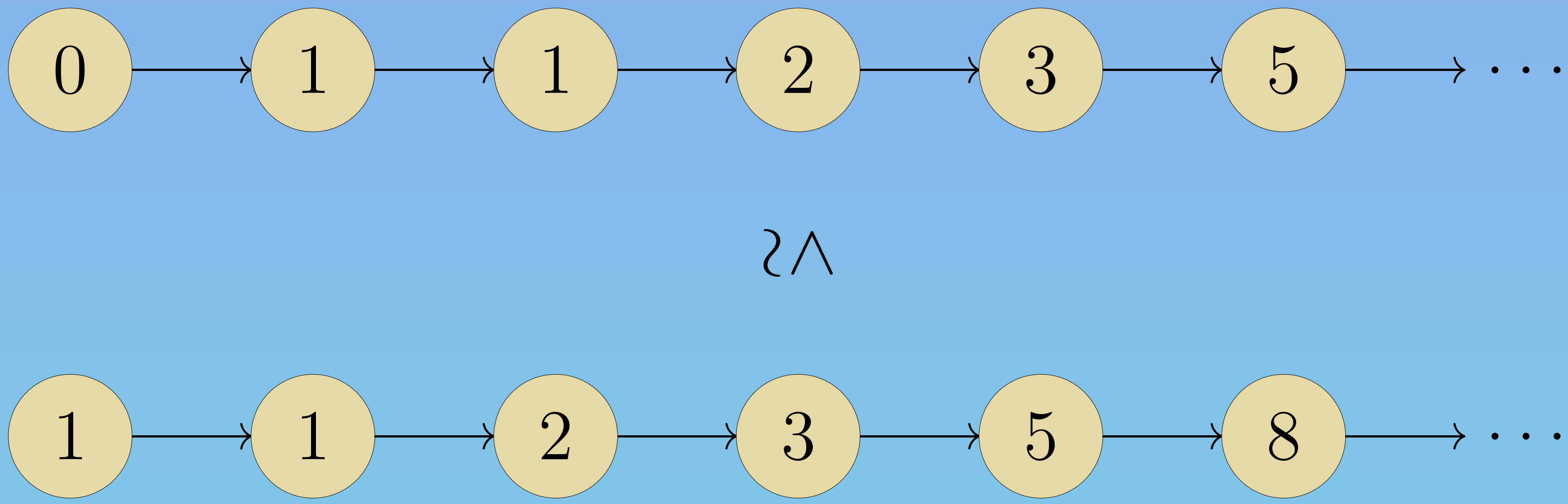

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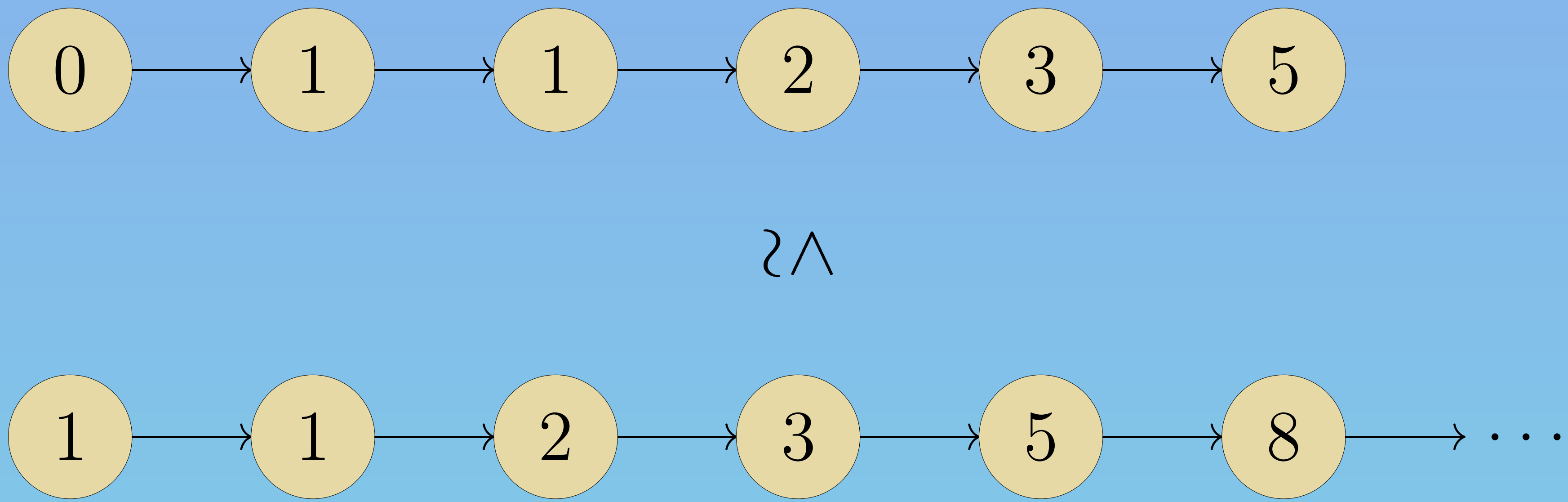

Related work: weak relators

[Thijs 1996, Baltag 2000]

Given a functor $F : \mathbf{Set} \to \mathbf{Set}$, a *weak relator extending F* is a functor $G : \mathbf{Rel} \to \mathbf{Rel}$ such that

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- $=_{FX} \subseteq G(=_X)$
- $R \subseteq S \Rightarrow GR \subseteq GS$.
- $GR \circ GS = G(R \circ S)$
- “functoriality”

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Theorem (Thijs). *Weak relators extending F are equivalent to lax relation liftings.*

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Thus, the difference between the two approaches is largely conceptual… but with some practical consequences.

Conceptual differences

Ordered functors

- Given: $\sqsubseteq$ and $\mathbf{Rel}(F)$

Weak relators

- Given: relators (lax relation liftings)

Conceptual differences

Ordered functors	Weak relators
• Given: $\sqsubseteq$ and $\mathbf{Rel}(F)$	• Given: relators (lax relation liftings)
• Derived: lax relation lifting	• Derived: $\sqsubseteq$ and $\mathbf{Rel}(F)$

Conceptual differences

Ordered functors	Weak relators
• Given: $\sqsubseteq$ and $\mathbf{Rel}(F)$	• Given: relators (lax relation liftings)
• Derived: lax relation lifting	• Derived: $\sqsubseteq$ and $\mathbf{Rel}(F)$
• Bisimulation is primitive	• Bisimulation is special case

Conceptual differences

Ordered functors

- Given: $\sqsubseteq$ and $\mathbf{Rel}(F)$
- Derived: lax relation lifting
- Bisimulation is primitive
- Emphasizes order-theoretic structure

Weak relators

- Given: relators (lax relation liftings)
- Derived: $\sqsubseteq$ and $\mathbf{Rel}(F)$
- Bisimulation is special case

Two-way similarity

Recall: $a \underline{\leftrightarrow} b \Leftrightarrow \exists$ bisimulation $R \,.\, aRb$.

Two-way similarity

Recall: $a \,\underline{\leftrightarrow}\, b \Leftrightarrow \exists R\,.\,aRb$ and $R,\ R^{\mathrm{op}}$ are simulations

Two-way similarity

Recall: $a \underline{\leftrightarrow} b \Leftrightarrow \exists R \,.\, aRb$ and R, R^{op} are simulations

Recall: $a \eqsim b \Leftrightarrow a \lesssim b$ and $b \lesssim a$.

Two-way similarity

Recall: $a \,\underline{\leftrightarrow}\, b \Leftrightarrow \exists R \,.\, aRb$ and $R,\ R^{\mathrm{op}}$ are simulations

Recall: $a \precsim\!\succsim b \Leftrightarrow \exists R,\ S \,.\, aRb,\ bSa$ and R, S are simulations.

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Recall: $a \precsim\succsim b \Leftrightarrow \exists R, \; S \,.\, aRb, \; bSa$ and R, S are simulations.

Note: Clearly, if $x \underline{\leftrightarrow} y$ then $x \precsim\succsim y$.

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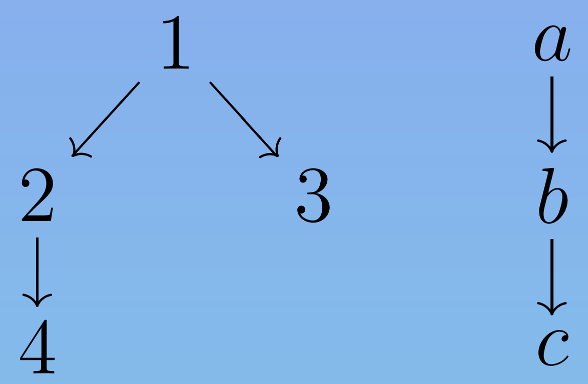
Recall: $a \precsim\!\succsim b \Leftrightarrow \exists R,\ S \,.\, aRb,\ bSa$ and R, S are simulations.

Note: Clearly, if $x \,\underline{\leftrightarrow}\, y$ then $x \precsim\!\succsim y$.

Question: if $x \precsim\!\succsim y$ then $x \,\underline{\leftrightarrow}\, y$?

Two-way similarity

The condition is non-trivial.



A counterexample.

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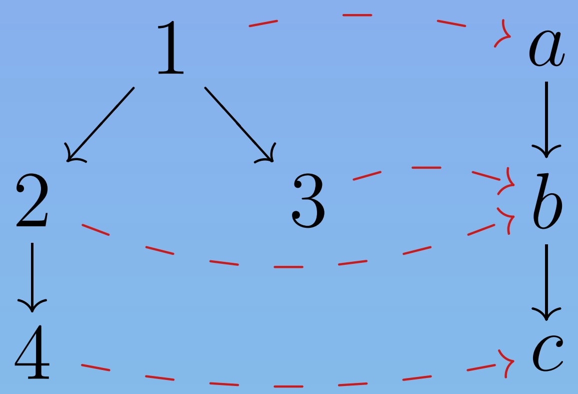

$$1 \precsim a.$$

Two-way similarity

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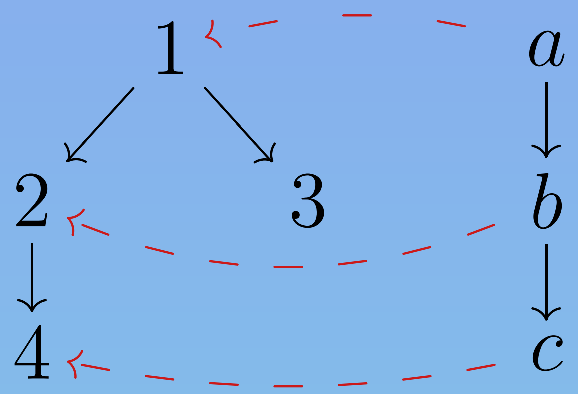

$$a \precsim 1.$$

Two-way similarity

Theorem. *Suppose that* $\sqsubseteq$ *satisfies*

$$\mathbf{Rel}_{\sqsubseteq}(F)(R_1) \cap \mathbf{Rel}_{\sqsubseteq^{\mathrm{op}}}(F)(R_2) \subseteq \mathbf{Rel}(F)(R_1 \cap R_2).$$

Then $\eqsim$ *coincides with* $\underline{\leftrightarrow}$.

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in terms of relators.

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in terms of relators.

How to eliminate $\sqsubseteq^{\mathrm{op}}$?

DCPOs

Let $F : \mathbf{Set} \to \mathbf{Set}$ and $\sqsubseteq : \mathbf{Set} \to \mathbf{PreOrd}$ be given.

Let $\zeta : Z \to FZ$ be the final F-coalgebra and $\lesssim$ its similarity order.

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Question: When is $\lesssim$ on Z a DCPO?

DCPOs

Let $F : \mathbf{Set} \to \mathbf{Set}$ and $\sqsubseteq : \mathbf{Set} \to \mathbf{PreOrd}$ be given.

Let $\zeta : Z \to FZ$ be the final F-coalgebra and $\lesssim$ its similarity order.

Question: When is $\lesssim$ on Z a DCPO?

When does every directed subset of Z have a join?

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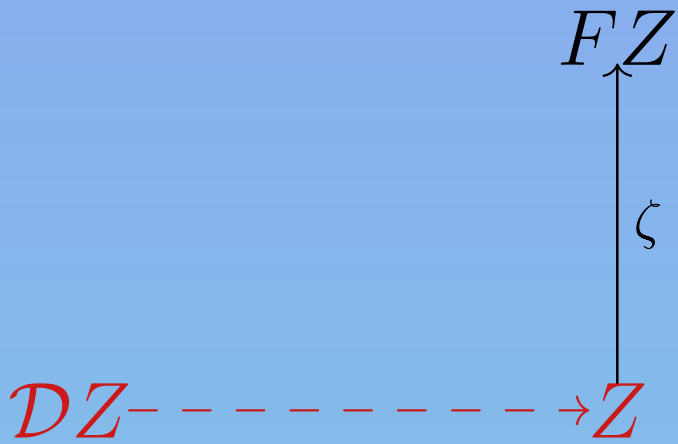
Fact: A preorder X is a DCPO iff the unit

$$\eta_X \colon X \longrightarrow \mathcal{D}X$$
$$x \longmapsto {\downarrow} x$$

has a left adjoint $\bigvee\colon\mathcal{D}X\to X$.

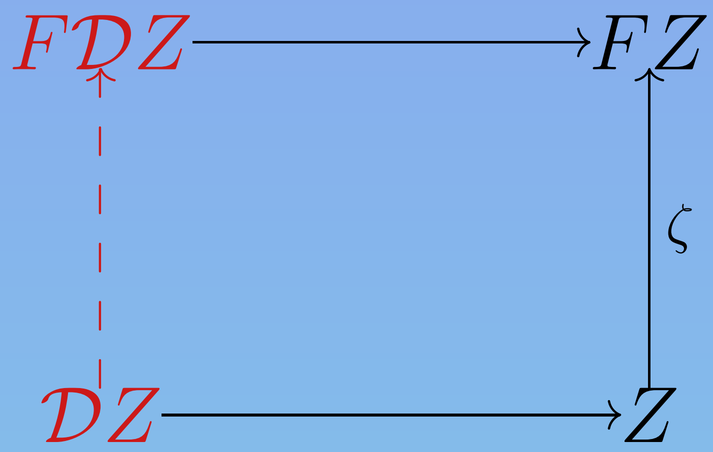
DCPOs continued

So, we want to define a left adjoint $\mathcal{D}Z \to Z$ to $z \mapsto \downarrow z$.



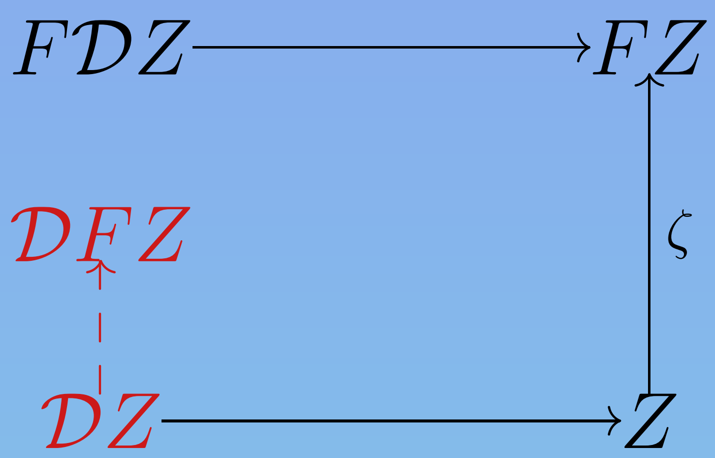
DCPOs continued

We can do this by defining $\mathcal{D}Z \to F\mathcal{D}Z$.



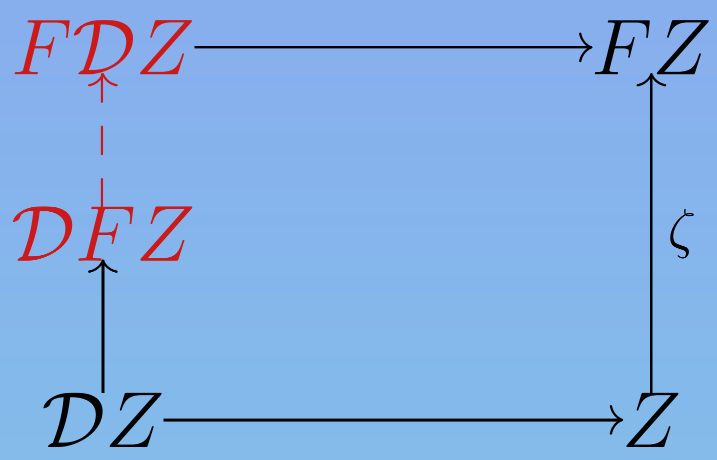
DCPOs continued

The map $\zeta : Z \to FZ$ is monotone, so we have
$\mathcal{D}\zeta : \mathcal{D}Z \to \mathcal{D}FZ$.



DCPOs continued

We acquire $\mathcal{D}FZ \rightarrow F\mathcal{D}Z$ by imposing a distributive law.



Distributive law

We suppose that $F : \mathbf{Set} \to \mathbf{Set}$ is a functor with order $\sqsubseteq$ with a natural transformation

$$\begin{array}{ccc} \mathbf{PreOrd} & \xrightarrow{\mathcal{D}} & \mathbf{PreOrd} \\ {\scriptstyle \mathbf{Rel}_{\sqsubseteq}(F)} \downarrow & \overset{\tau}{\Nearrow} & \downarrow {\scriptstyle \mathbf{Rel}_{\sqsubseteq}(F)} \\ \mathbf{PreOrd} & \xrightarrow[\mathcal{D}]{} & \mathbf{PreOrd} \end{array}$$

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\end{array}$$

satisfying the diagrams

$$\begin{array}{ccc}
F & \overset{\eta_F}{\Longrightarrow} & \mathcal{D}F \\
& {\scriptstyle F\eta}\searrow\!\!\!\searrow & \Downarrow{\scriptstyle \tau} \\
& & F\mathcal{D}
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{D}^2F & \overset{\mathcal{D}\tau}{\Longrightarrow} & \mathcal{D}F\mathcal{D} & \overset{\tau_{\mathcal{D}}}{\Longrightarrow} & F\mathcal{D}^2 \\
{\scriptstyle \mu_F}\Downarrow & & & & \Downarrow{\scriptstyle F\mu} \\
\mathcal{D}F & & \overset{}{\underset{\tau}{\Longrightarrow}} & & F\mathcal{D}
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Distributive law

Theorem. *In this situation, the final coalgebra* $\langle Z, \zeta \rangle$ *is a DCPO under the similarity order* $\lesssim$.

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The distributive law $\tau : \mathcal{D}F \to F\mathcal{D}$ can also be constructed inductively on the structure of F.

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- We derive lax relation lifting and a notion of simulation.
- We find this development more natural than taking lax relation lifting as primitive.
- We find a sufficient condition for $\underline{\leftrightarrow} = \eqsim$.
- A distributive law ensures that $\lesssim$ on the final coalgebra is a DCPO.
- When is $\lesssim$ an algebraic DCPO?