The Formal Dual of Birkhoff's Completeness Theorem

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Outline

I. Coequations

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II. Conditional coequations

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III. Horn coequations

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IV. Some co-Birkhoff type theorems (again)

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Coequations

Let $U \dashv H$ and $C \in \mathcal{C}$ be injective with respect to $\mathcal{S}$-morphisms.

A *coequation over C* is an $\mathcal{S}$-morphism $P \rightarrowtail UHC$ in $\mathcal{C}$.

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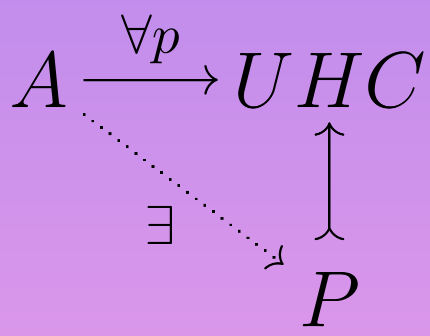
We say $\langle A, \alpha \rangle \models_C P$ just in case for every homomorphism $p : \langle A, \alpha \rangle \to HC$, we have $\mathsf{Im}(p) \leq P$.



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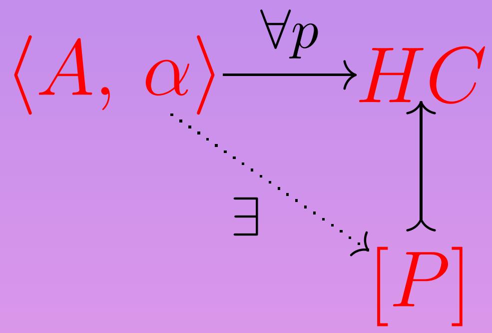

Here, $[P]$ is the largest subcoalgebra of HC contained in P.

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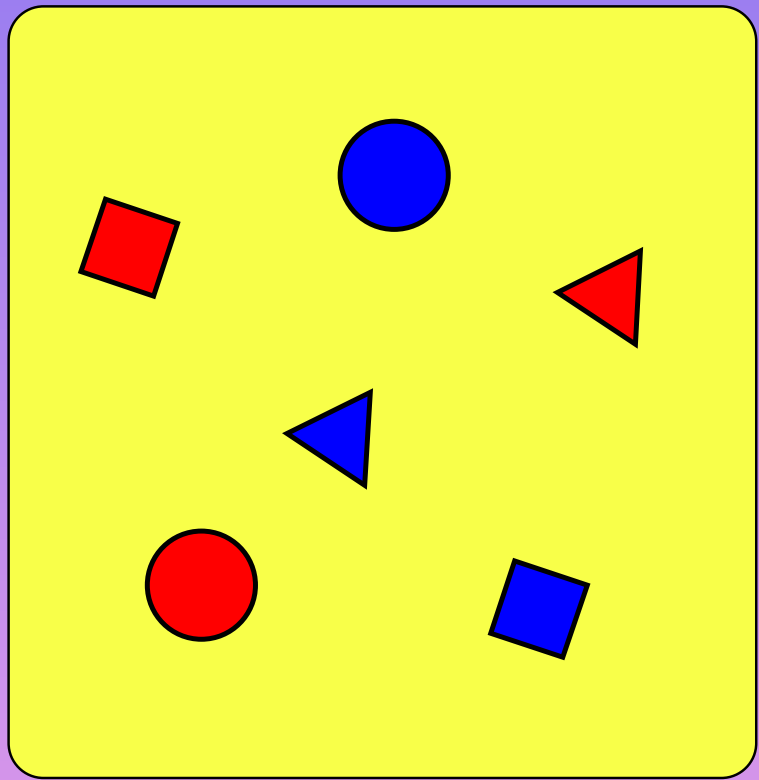
We say $\langle A,\, \alpha\rangle \models_C P$ just in case for every homomorphism $p\colon \langle A,\, \alpha\rangle \to HC$, we have $\mathsf{Im}(p) \leq P$.

Thus, $\langle A,\, \alpha\rangle \models_C P$ iff $\langle A,\, \alpha\rangle \in \mathbf{Proj}([P])$, i.e.,

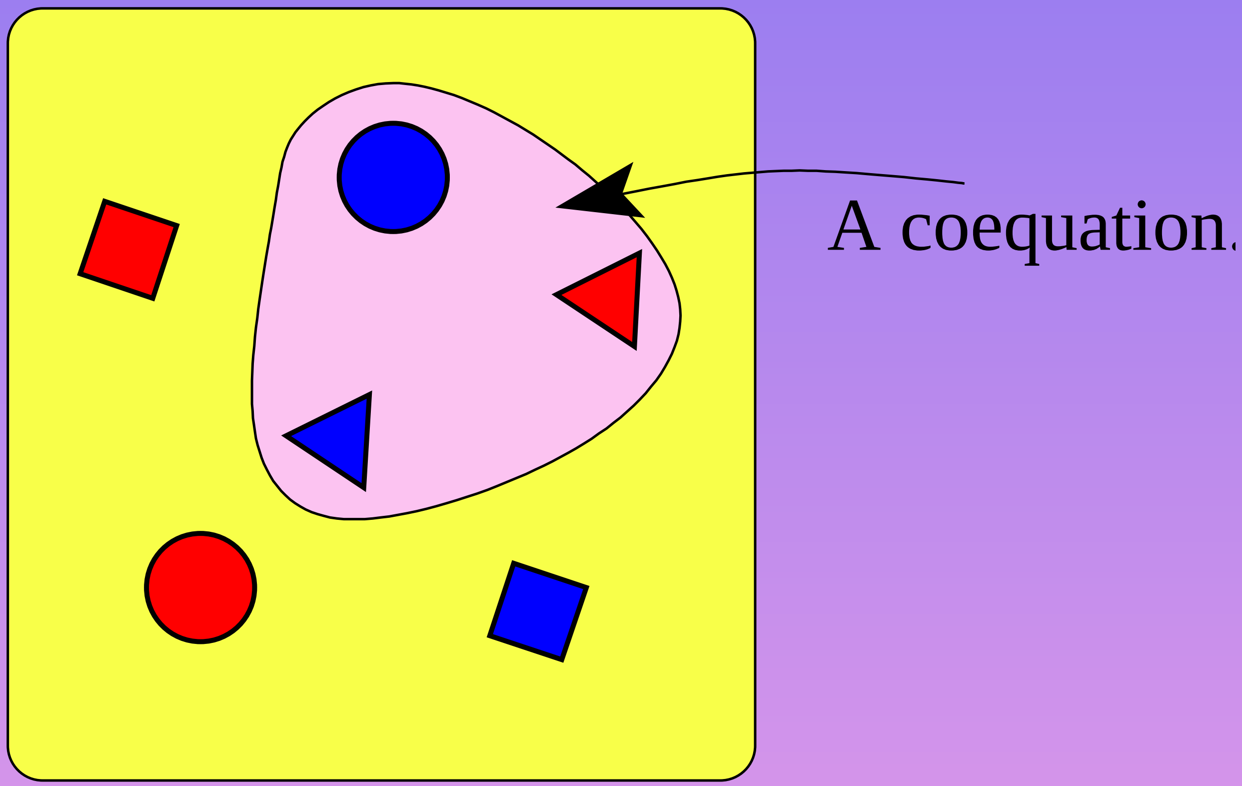
$$\mathsf{Hom}(\langle A,\, \alpha\rangle, HC) \cong \mathsf{Hom}(\langle A,\, \alpha\rangle, [P]).$$

Example



The cofree coalgebra $H2$

Example



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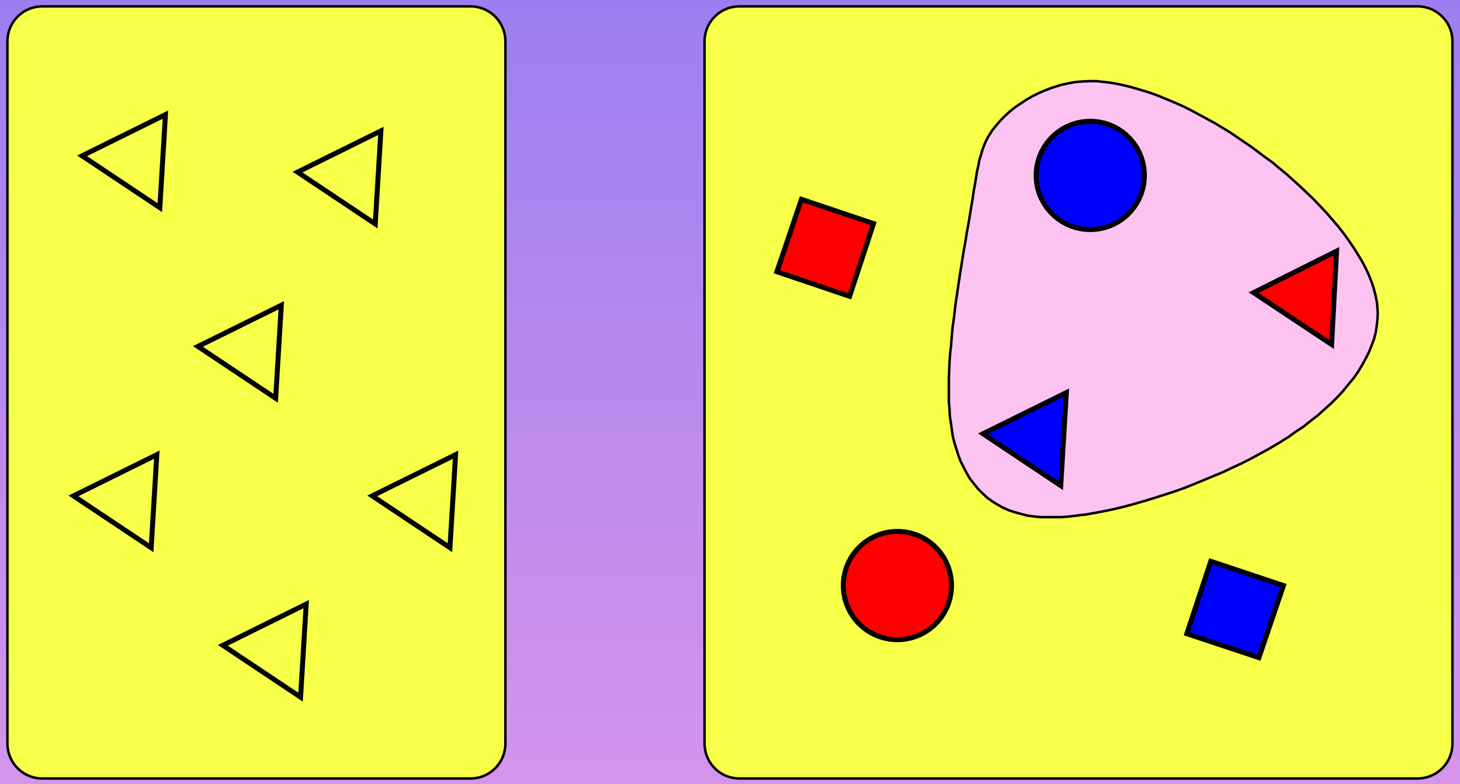
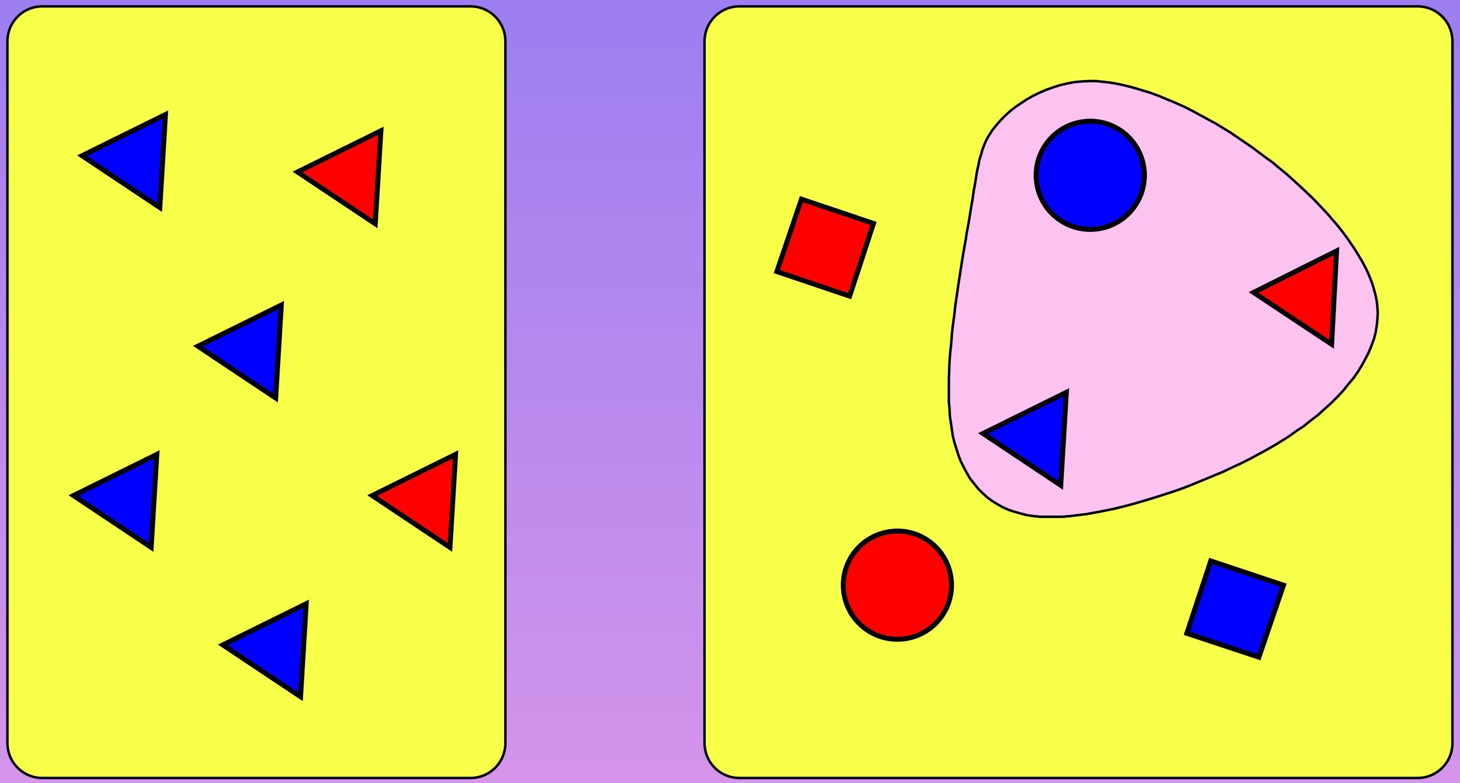


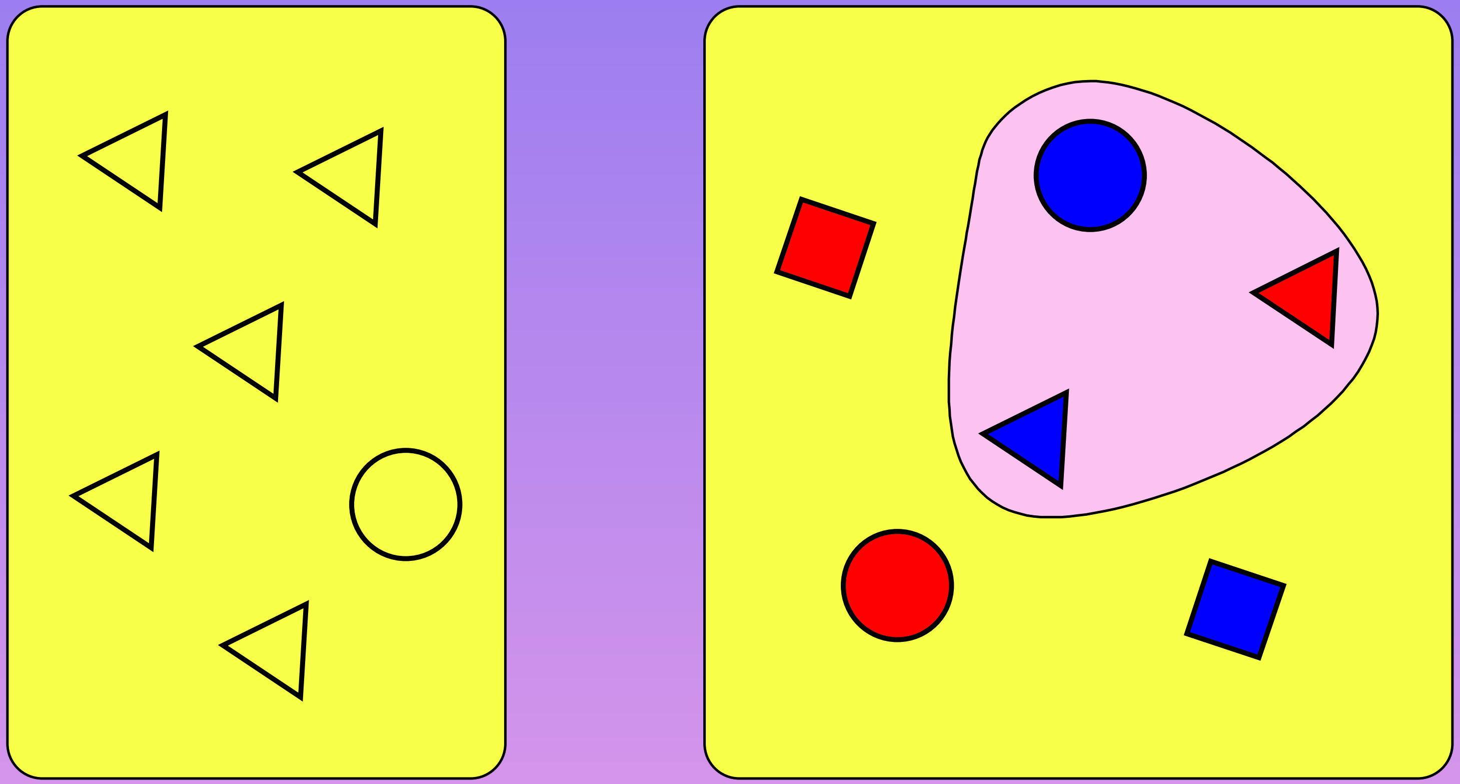
This coalgebra satisfies P.

Example



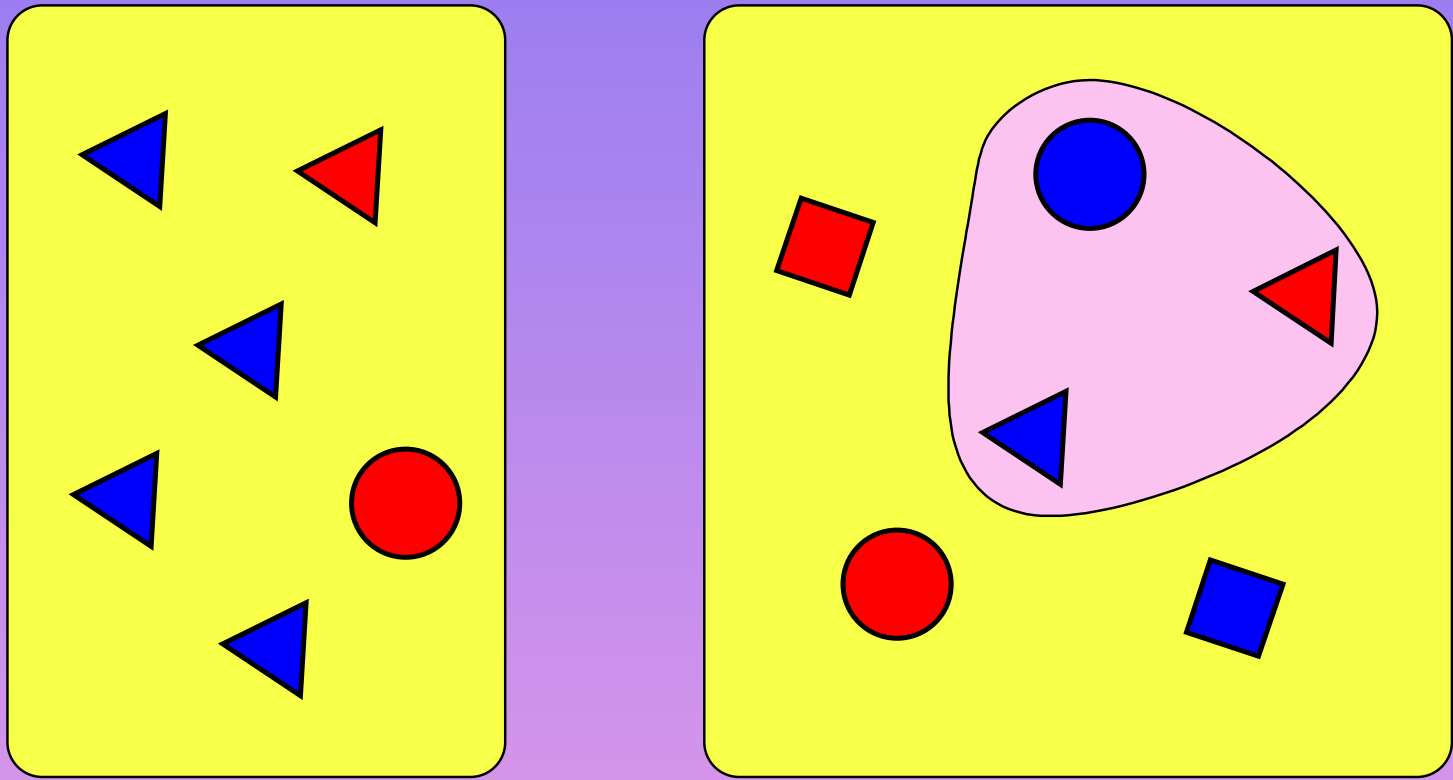
Under any coloring, the elements of the coalgebra map to elements of P.

Example



This coalgebra doesn't satisfy P.

Example



If we paint the circle red, it isn't mapped to an element of P.

Comparing coequations and equations

Algebras	Coalgebras
Projective set of variables X	Injective set of colors C

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$q : FX \twoheadrightarrow \langle Q, \nu \rangle$	$i : [P] \rightarrowtail HC$

Comparing coequations and equations

Algebras	Coalgebras
Projective set of variables X	Injective set of colors C
Set of equations	Coequation
$E \rightrightarrows UFX$	$P \rightarrowtail UHC$
$q : FX \twoheadrightarrow \langle Q, \nu \rangle$	$i : [P] \rightarrowtail HC$
$\models$ as q-injective	$\models$ as i-projective

Conditional coequations

Let $P,\, Q \leq UHC$.
We write $\langle A,\, \alpha\rangle \models_C P \Rightarrow Q$ just in case, for every $p\colon\langle A,\, \alpha\rangle\to HC$ such that $\mathsf{Im}(p) \leq P$, we have $\mathsf{Im}(p) \leq Q$.

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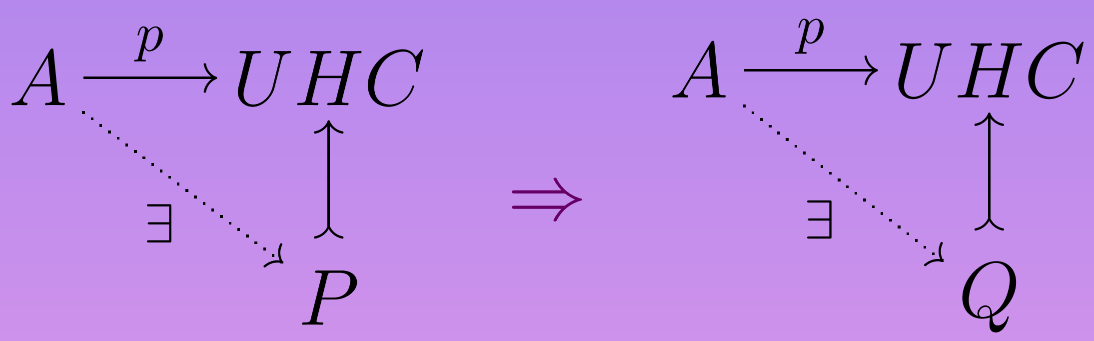


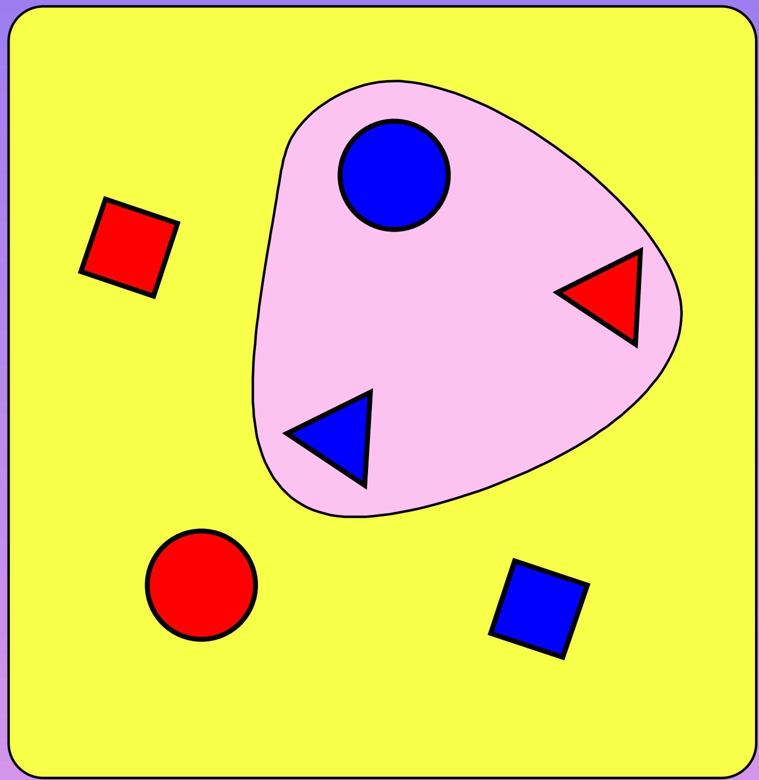
Conditional coequations

Let $P, Q \leq UHC$.
We write $\langle A, \alpha\rangle \models_C P \Rightarrow Q$ just in case, for every $p : \langle A, \alpha\rangle \to HC$ such that $\mathsf{Im}(p) \leq P$, we have $\mathsf{Im}(p) \leq Q$.

$\langle A, \alpha\rangle \models P \Rightarrow Q$ just in case every homomorphism $\langle A, \alpha\rangle \to [P]$ factors through $[Q]$, i.e.,

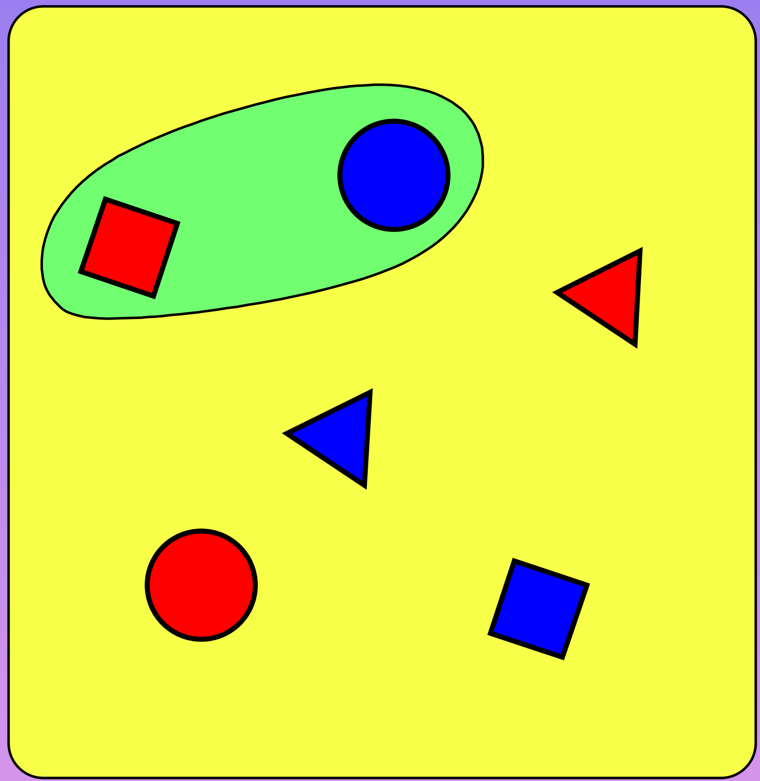
$$\mathsf{Hom}(\langle A, \alpha\rangle, [P]) \cong \mathsf{Hom}(\langle A, \alpha\rangle, [Q]).$$

Example



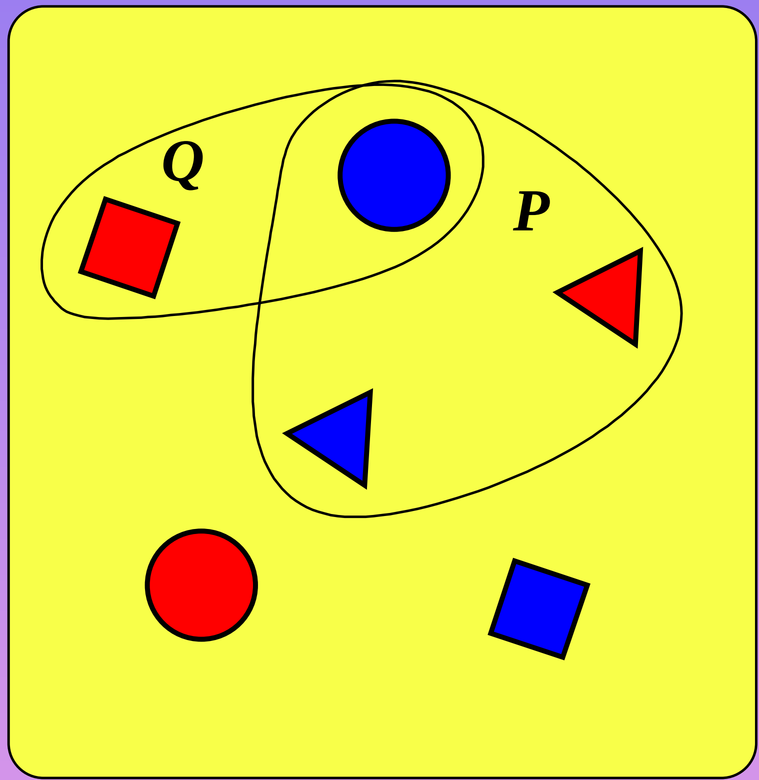
Recall our coequation P.

Example



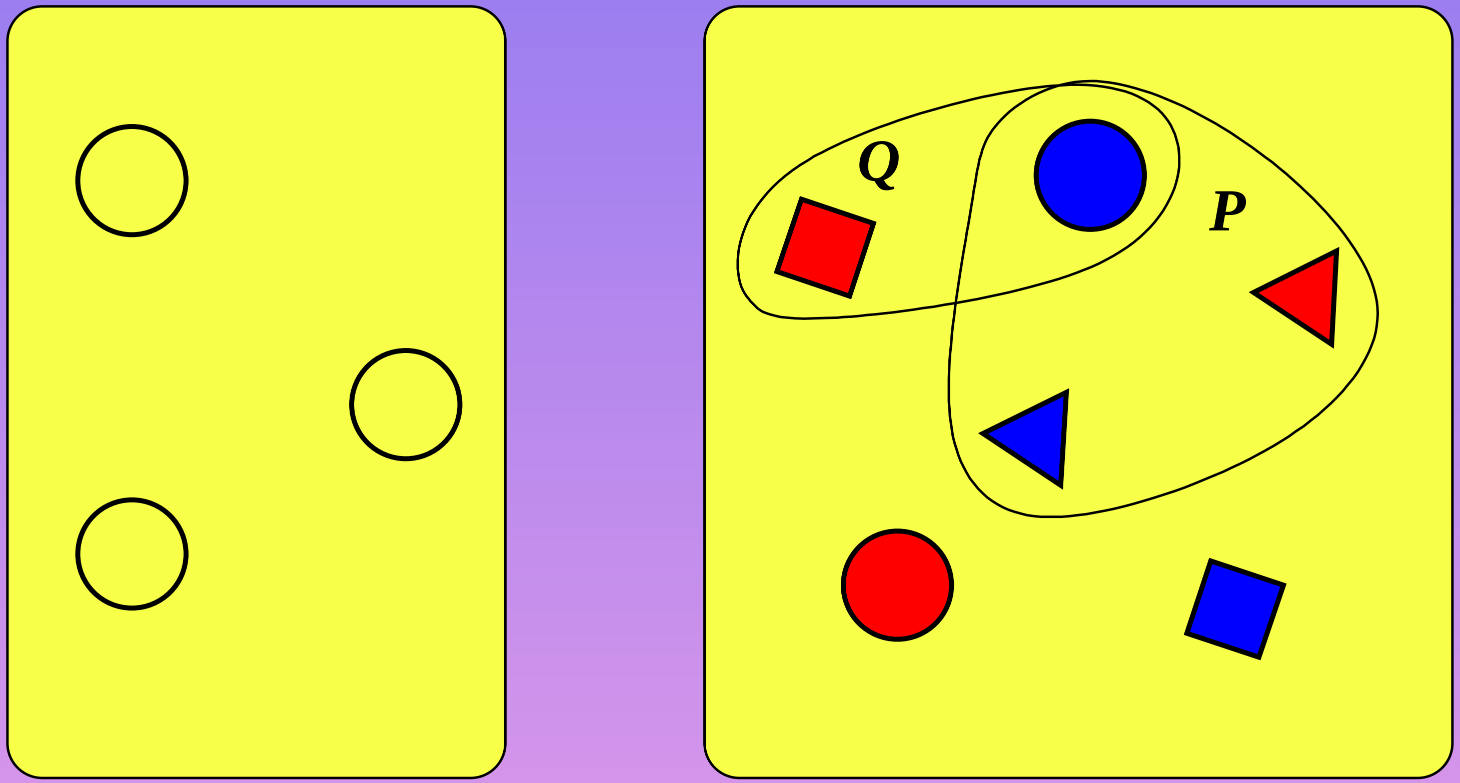
Let Q be the coequation above.

Example



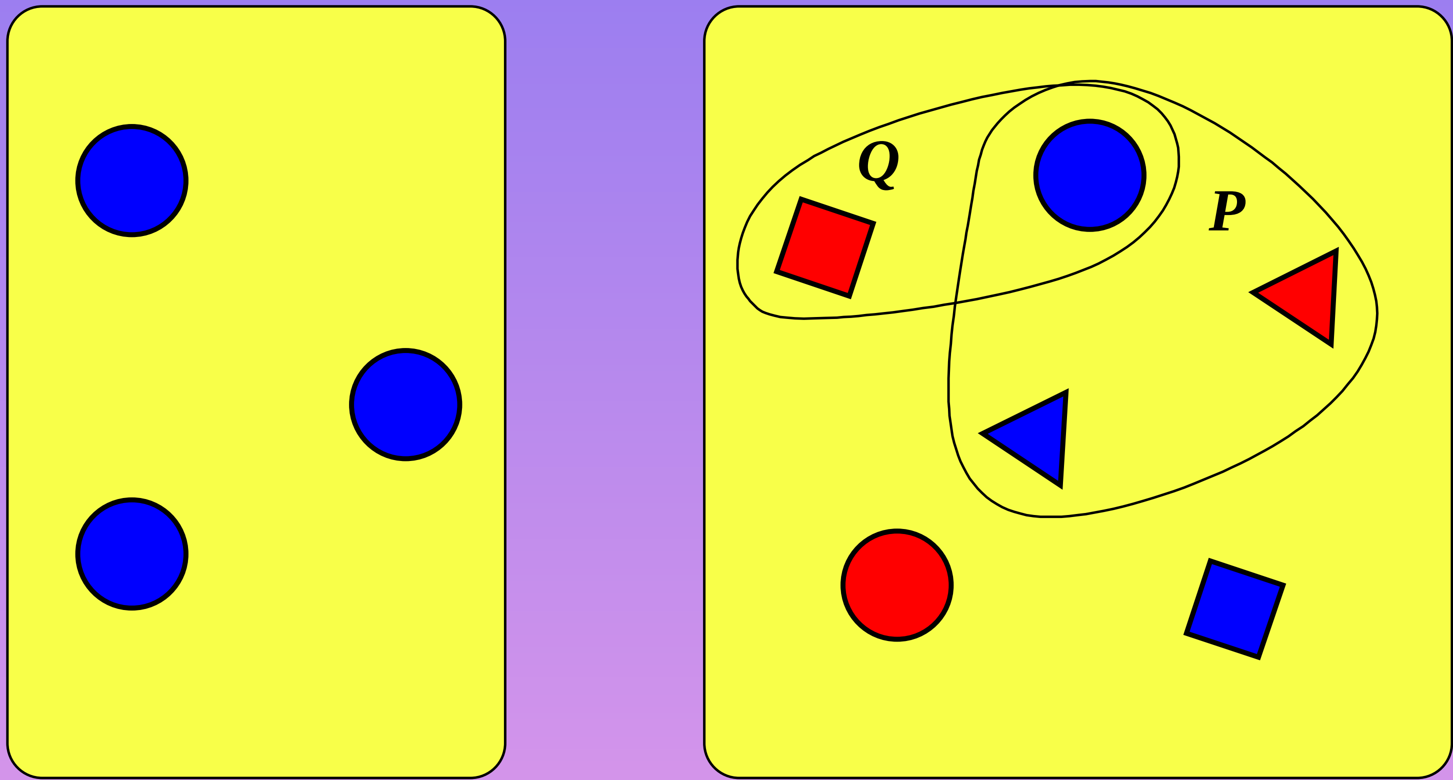
And consider the "conditional coequation" $P \Rightarrow Q$.

Example



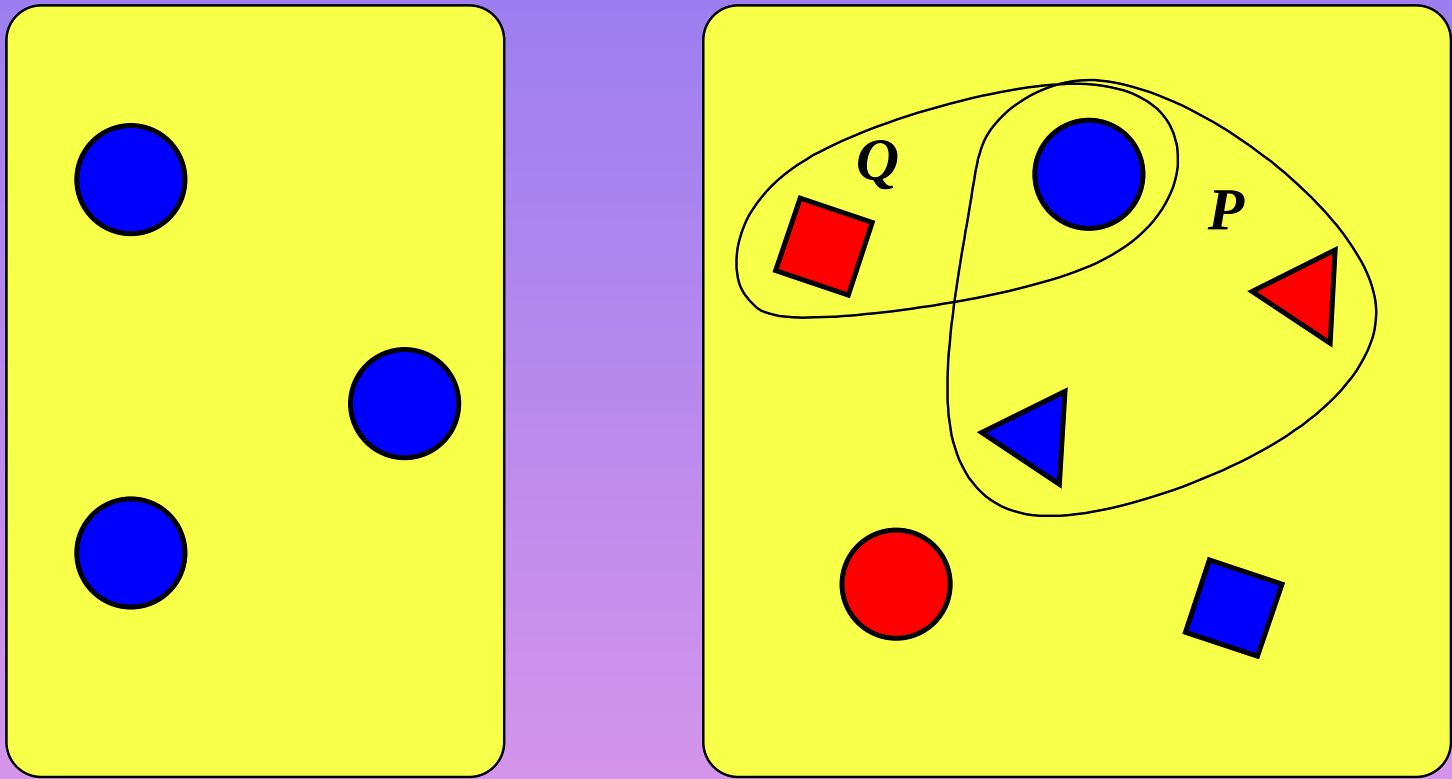
This coalgebra satisfies $P \Rightarrow Q$.

Example



However we paint it so that it factors through P, it also factors through Q.

Example



(It also satisfies $Q \Rightarrow P$.)

Dualizing negations

Let $P \leq UHC$.

We write $\langle A, \alpha\rangle \models_C \overline{P}$ just in case for every $p\colon A \to C$, it is not the case $\mathsf{Im}(\widetilde{p}) \leq P$.

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Equivalently, there is no homomorphism $\langle A,\, \alpha\rangle \to [P]$, i.e.,

$$\mathsf{Hom}(\langle A,\, \alpha\rangle, [P]) = \emptyset.$$

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No matter how we paint A, there is some element $a \in A$ that doesn't land in P.

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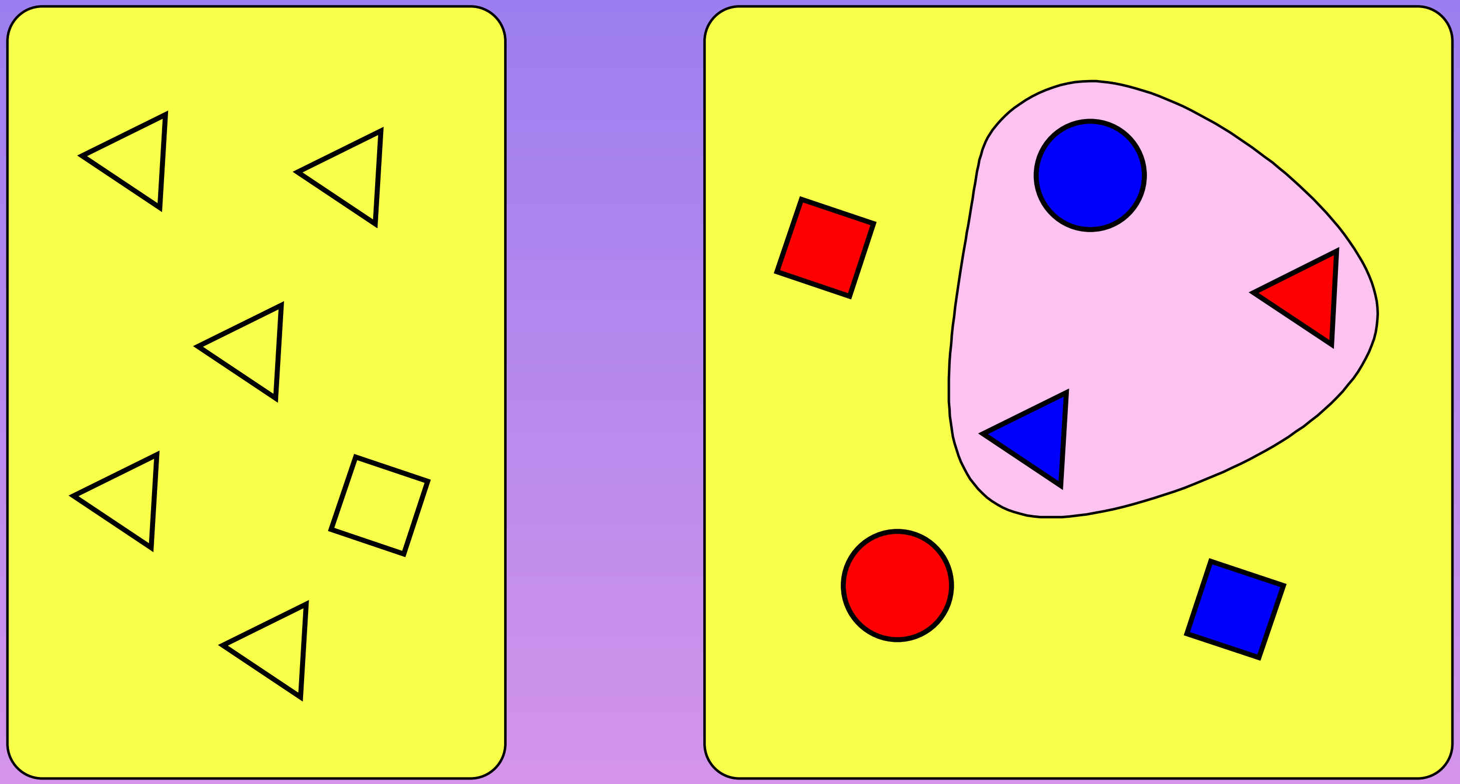
No matter how we paint A, there is some element $a \in A$ that doesn't land in P.

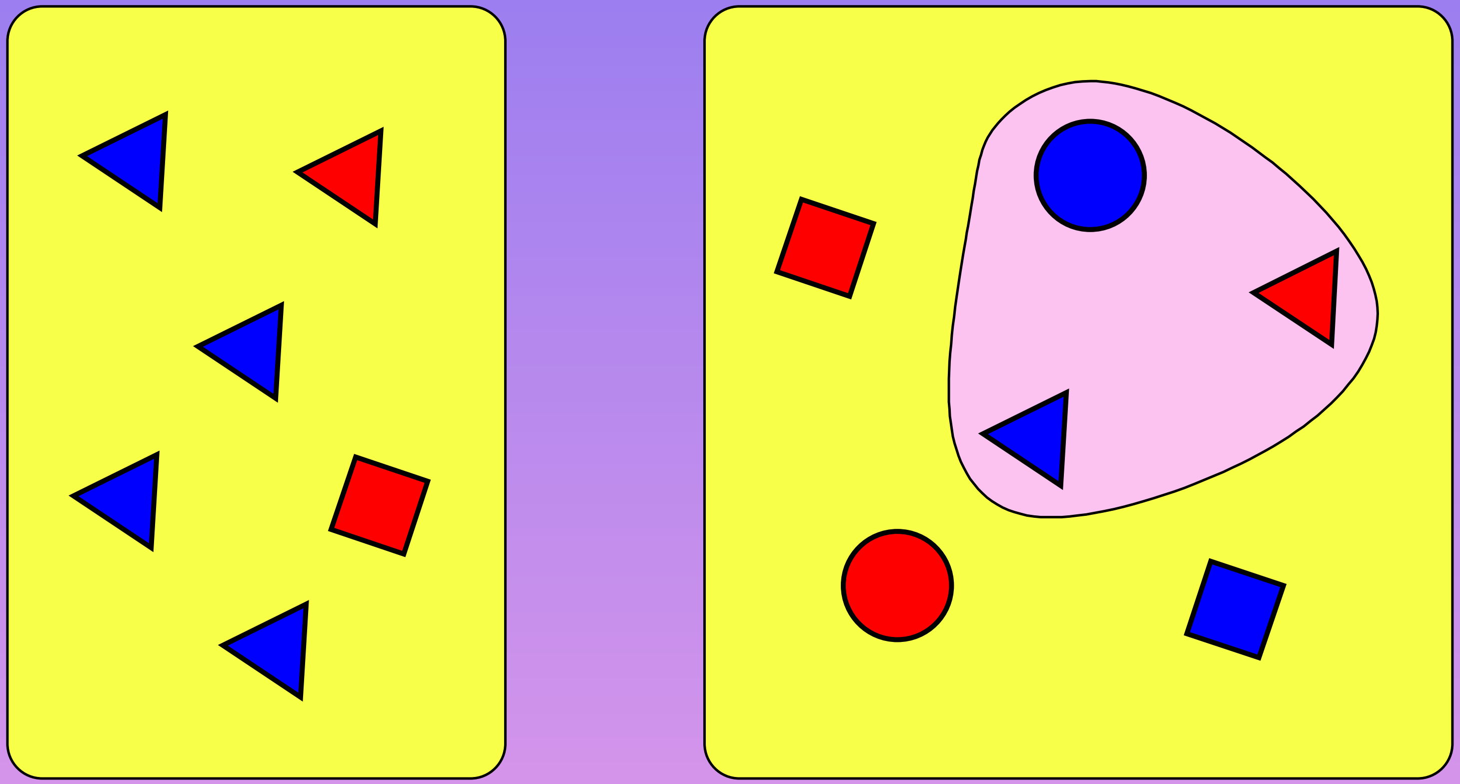
Note: This does not mean that $\langle A, \alpha \rangle \models \neg P$! "Something in A does not land in P," is not the same as, "Everything in A does not land in P."

Example



The coalgebra on the left satisfies $\overline{P}$.

Example



No matter how we paint it, the square does not land in P

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Some co-Birkhoff-type theorems

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$$\mathsf{Th}\,\mathbf{V} = \{P \rightarrowtail UHC \mid \mathbf{V} \models_C P\}$$

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$$\mathsf{Horn}\,\mathbf{V} = \mathsf{Imp}\,\mathbf{V} \cup \{\overline{P}^C \mid \mathbf{V} \models_C \overline{P}\}$$

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Further, let $\mathsf{Mod}\,S$ denote the models of S for S a class of coequations, conditional coequations or Horn coequations.

Some co-Birkhoff-type theorems

Theorem (Birkhoff covariety theorem).

$$\mathsf{Mod}\,\mathsf{Th}\,\mathbf{V} = \mathcal{SH}\Sigma\mathbf{V}$$

Theorem (Quasi-covariety theorem).

$$\mathsf{Mod}\,\mathsf{Imp}\,\mathbf{V} = \mathcal{H}\Sigma\mathbf{V}$$

Theorem (Horn covariety theorem).

$$\mathsf{Mod}\,\mathsf{Horn}\,\mathbf{V} = \mathcal{H}\Sigma^{+}\mathbf{V}$$

Birkhoff's deduction theorem

Fix a set X of variables and let E be a set of equations over X. E is deductively closed just in case E satisfies the following:

(i) $x = x \in E$;

(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;

(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;

(iv) $t_1^i = t_2^i \in E$ and $f \in \Sigma \Rightarrow f(\vec{t_1}) = f(\vec{t_2}) \in E$;

(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

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Items (i) – (iv) ensure that E is a congruence and hence uniquely determines a quotient of FX.

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Item (v) ensures that E is a stable $\mathbb{P}$-algebra, i.e., closed under substitutions.

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Let $\mathsf{Ded} : \mathsf{Rel}(UFX) \to \mathsf{Rel}(UFX)$ be the closure operation taking a set E of equations over X to its deductive closure. We can decompose Ded into two closure operators.

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The first takes E to the congruence it generates.

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The second closes it under substitution of terms for variables.

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Theorem (Birkhoff completeness theorem). *For any* $E \in \mathsf{Rel}(UFX)$, $\mathsf{Th}\,\mathsf{Mod}(E) = \mathsf{Ded}(E)$

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Compare this to the variety theorem.

Theorem (Birkhoff variety theorem).

$$\mathsf{Mod}\,\mathsf{Th}\,\mathbf{V} = \mathcal{HSP}\mathbf{V}$$

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$\mathsf{Th}\,\mathsf{Mod}(E)$ satisfies the following fixed point description.

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Recall that sets of equations correspond to coequations, so this is an appropriate dualization.

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Recall that sets of equations correspond to coequations, so this is an appropriate dualization.

A generating coequation gives a measure of the "coequational commitment" of $\mathbf{V}$.

Dualizing deductive closure

Theorem (Birkhoff completeness theorem). *For any* $E \in \mathsf{Rel}(UFX)$, $\mathsf{Th}\,\mathsf{Mod}(E) = \mathsf{Ded}(E)$

To dualize Ded, we consider again its components.

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Projective set of variables X	Injective set of colors C
Set of equations	Coequation
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Closure under substitution	Greatest endo-invariant sub-object

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The modal operator $\Box$

Let $P, Q \rightarrowtail A$ be given. We write $P \vdash Q$ if there is a map $P \rightarrow Q$ such that the diagram below commutes.

$$\begin{array}{ccc} P & \longrightarrow & Q \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

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In fact, $P \rightarrowtail Q$ is necessarily an $\mathcal{S}$-morphism.

The modal operator $\Box$

Let $\Box : \mathsf{Sub}(UHC) \rightarrow \mathsf{Sub}(UHC)$ be the composite $U[-]$. In other terms, $\Box$ is a comonad taking a coequation P to the largest subcoalgebra $\langle A,\, \alpha \rangle$ of HC such that $A \leq P$.

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As is well-known, if Γ preserves pullbacks of $\mathcal{S}$-morphisms, then $\Box$ is an **S4** operator.

(i) If $P \vdash Q$ then $\Box P \vdash \Box Q$;

(ii) $\Box P \vdash P$;

(iii) $\Box P \vdash \Box \Box P$;

(iv) $\Box(P \to Q) \vdash \Box P \to \Box Q$;

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(i) follows from functoriality.

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(ii) and (iii) are the counit and comultiplication of the comonad.

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(iv) follows from the fact that $U : \mathcal{E}_\Gamma \to \mathcal{E}$ preserves finite meets.

The modal operator $\Box$

(i) If $P \vdash Q$ then $\Box P \vdash \Box Q$;

(ii) $\Box P \vdash P$;

(iii) $\Box P \vdash \Box\,\Box\,P$;

(iv) $\Box(P \to Q) \vdash \Box P \to \Box Q$;

Proof.

$$\frac{P \to Q \vdash P \to Q}{(P \to Q) \wedge P \vdash Q}$$

By the counit of adjunction $- \wedge P \dashv P \to -$. $\ggg$

The modal operator $\Box$

(i) If $P \vdash Q$ then $\Box P \vdash \Box Q$;

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(iv) $\Box(P \to Q) \vdash \Box P \to \Box Q$;

Proof.

$$\frac{(P \to Q) \wedge P \vdash Q}{\Box((P \to Q) \wedge P) \vdash \Box Q}$$

By (i). $\ggg$

The modal operator $\Box$

(i) If $P \vdash Q$ then $\Box P \vdash \Box Q$;

(ii) $\Box P \vdash P$;

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(iv) $\Box(P \to Q) \vdash \Box P \to \Box Q$;

Proof.

$$\frac{\Box((P \to Q) \wedge P) \vdash \Box Q}{\Box(P \to Q) \wedge \Box P \vdash \Box Q}$$

Because $\Box$ preserves meets. $\ggg$

The modal operator $\Box$

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(iii) $\Box P \vdash \Box\,\Box\, P$;

(iv) $\Box(P \to Q) \vdash \Box P \to \Box Q$;

Proof.

$$\frac{\Box(P \to Q) \wedge \Box P \vdash \Box Q}{\Box(P \to Q) \vdash \Box P \to \Box Q}$$

Again, by the adjunction $- \wedge P \dashv P \to -$. $\Box$

Invariant coequations

Let $f : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$ and $P \rightarrowtail A$ be given. We let $\exists_f P$ denote the image of the composite $P \rightarrowtail A \longrightarrow B$.

$$\begin{array}{ccc} P & \longrightarrow\!\!\!\!\!\rightarrow & \exists_f P \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Invariant coequations

Let $P \subseteq UHC$. We say that P is endomorphism-invariant just in case, for every "repainting"

$$p : UHC \longrightarrow C,$$

equivalently, every homomorphism $\widetilde{p} : HC \rightarrow HC$, we have

$$\exists_{\widetilde{p}} P \leq P.$$

Invariant coequations

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equivalently, every homomorphism $\widetilde{p}: HC \rightarrow HC$, we have

$$\exists_{c \in UHC}(\widetilde{p}(c) = x \wedge P(c)) \vdash P(x).$$

In other words, however we repaint HC, the elements of P are again (under this new coloring) elements of P.

Definition of $\boxslash$

Let $P \subseteq UHC$. Define

$$\mathcal{I}_P = \{Q \leq UHC \mid \forall p\colon HC \longrightarrow HC\,(\exists_p Q \leq P)\}.$$

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In particular, if $Q \in \mathcal{I}_P$, then $Q \vdash P$.

Definition of $\boxslash$

Let $P \subseteq UHC$. Define

$$\mathcal{I}_P = \{Q \leq UHC \mid \forall p : HC \longrightarrow HC(\exists_p Q \leq P)\}.$$

We define a functor $\boxslash : \mathsf{Sub}(UHC) \to \mathsf{Sub}(UHC)$ by

$$\boxslash P = \bigvee \mathcal{I}_P.$$

Then $\boxslash P$ is the greatest invariant subobject of UHC contained in P.

Definition of $\boxslash$

We define a functor $\boxslash : \mathsf{Sub}(UHC) \to \mathsf{Sub}(UHC)$ by

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That is, $\boxslash P$ satisfies the following:

- For all $p : HC \to HC$, $\exists_p \boxslash P \vdash \boxslash P$.

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We define a functor $\boxslash : \mathsf{Sub}(UHC) \to \mathsf{Sub}(UHC)$ by

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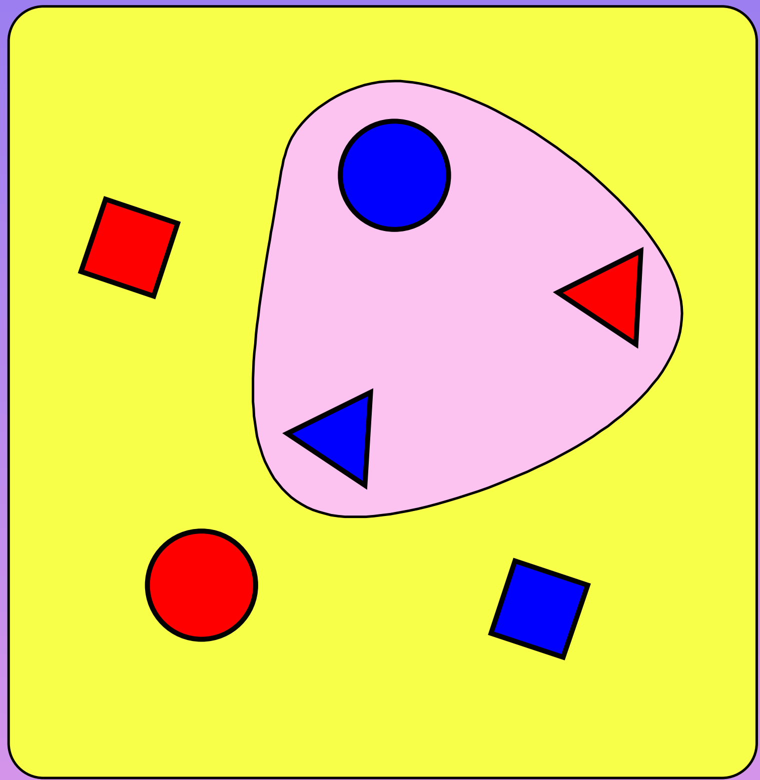
That is, $\boxslash P$ satisfies the following:

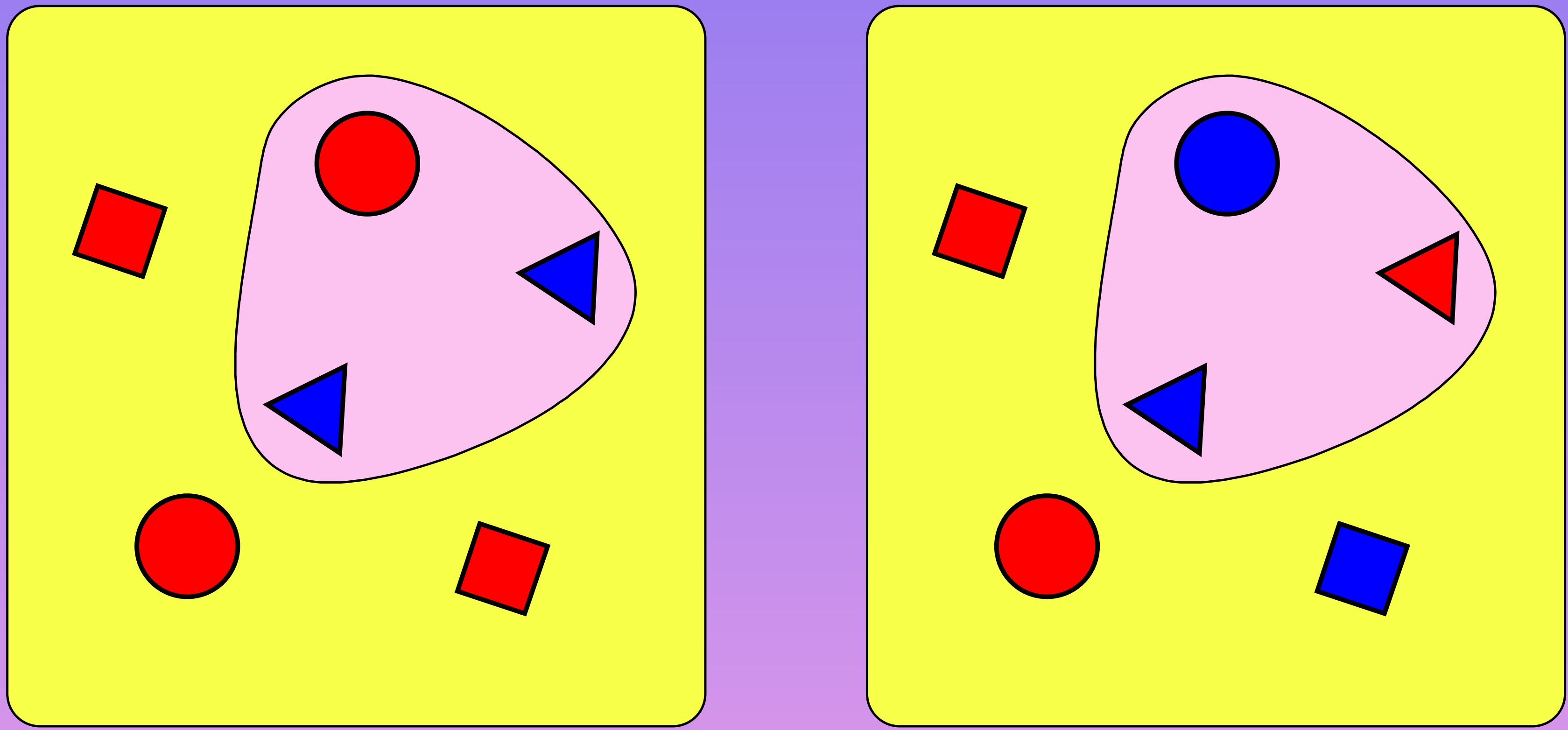
- For all $p : HC \to HC$, $\exists_p \boxslash P \vdash \boxslash P$.
- If $Q \vdash P$ and for all $p : HC \to HC$, $\exists_p Q \vdash Q$, then $Q \vdash \boxslash P$.

Example (cont.)



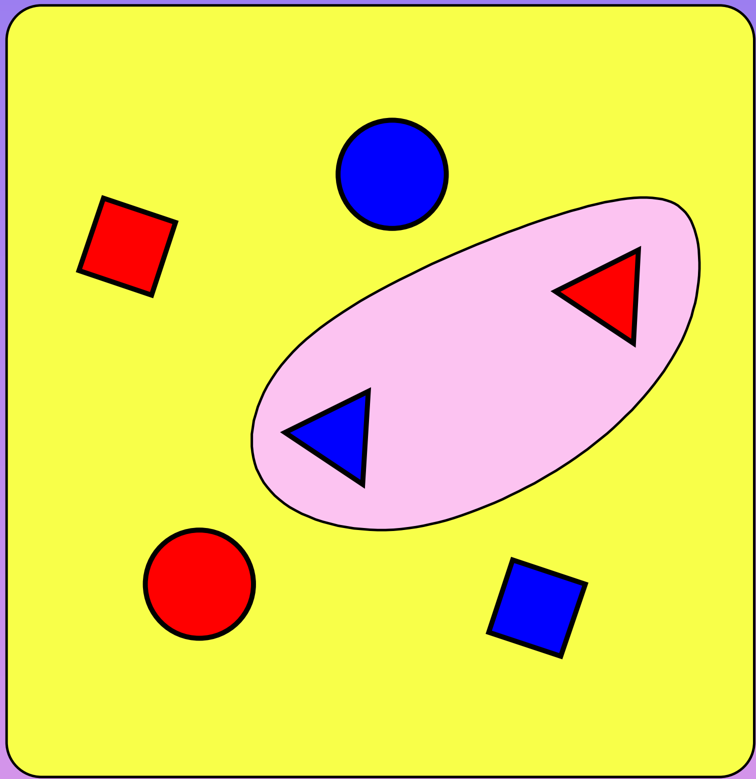
The coequation P.

Example (cont.)



P is not invariant.

Example (cont.)



The coequation $\boxslash P$.

$\boxslash$ is S4

One can show that $\boxslash$ is an **S4** operator.

(i) If $P \vdash Q$ then $\boxslash P \vdash \boxslash Q$;

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(i) - (iii) follow from the fact that $\boxslash$ is a comonad, as before.

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(iv) requires an argument that the meet of two invariant co-equations is again invariant. This is not difficult.

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The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ *iff* $\langle A, \alpha \rangle \models \Box P$.

The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ *iff* $\langle A, \alpha \rangle \models \square P$.

Lemma. $\langle A, \alpha \rangle \models P$ *iff* $\langle A, \alpha \rangle \models \boxslash P$.

The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ *iff* $\langle A, \alpha \rangle \models \Box \boxslash P$.

Lemma. $[\boxslash P] \models P$.

The invariance theorem

Lemma. $\langle A, \alpha\rangle \models P$ *iff* $\langle A, \alpha\rangle \models \Box \boxslash P$.

Lemma. $[\boxslash P] \models P$.

Lemma. $\Box \boxslash P \leq \boxslash \Box P$, *i.e., if* P *is invariant,then so is* $\Box P$.

The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ *iff* $\langle A, \alpha \rangle \models \Box \boxslash P$.

Lemma. $[\boxslash P] \models P$.

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Theorem. $\mathsf{Gen}\,\mathsf{Mod}\,P = \Box \boxslash P$.

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Theorem. $\mathsf{Gen}\,\mathsf{Mod}\,P = \Box \boxslash P$.

Proof. From the above, we see that $\mathsf{Mod}\,P \models \Box \boxslash P$.

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Proof. From the above, we see that $\mathsf{Mod}\,P \models \Box \boxslash P$. Suppose that $\mathsf{Mod}\,P \models Q$. Then $[\boxslash P] \models Q$. ⋙

The invariance theorem

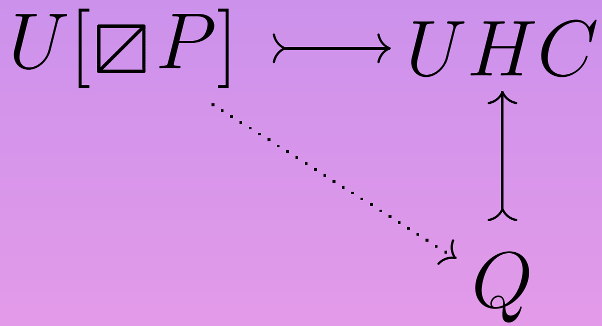
Lemma. $\langle A, \alpha \rangle \models P$ *iff* $\langle A, \alpha \rangle \models \square \boxslash P$.

Lemma. $[\boxslash P] \models P$.

Lemma. $\square \boxslash P \leq \boxslash \square P$.

Theorem. $\mathsf{Gen}\,\mathsf{Mod}\,P = \square \boxslash P$.

Proof. From the above, we see that $\mathsf{Mod}\,P \models \square \boxslash P$. Suppose that $\mathsf{Mod}\,P \models Q$. Then $[\boxslash P] \models Q$. Hence:



The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ *iff* $\langle A, \alpha \rangle \models \Box \boxslash P$.

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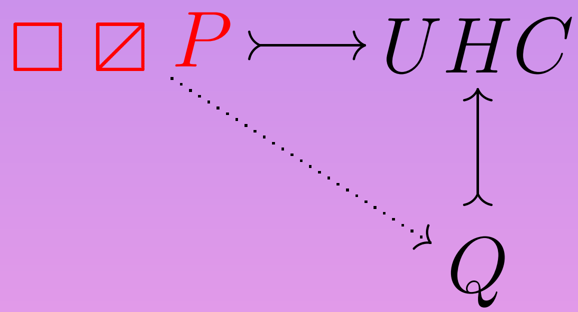

That is, $\Box \boxslash P \vdash Q$. $\Box$

Commutativity of $\Box$, $\boxslash$

As we saw (without proof),

Lemma. $\Box \boxslash P \leq \boxslash \Box P$.

That is, the greatest subcoalgebra of an endomorphism invariant predicate is itself invariant.

Commutativity of $\square$, $\boxslash$

As we saw (without proof),

Lemma. $\square \boxslash P \leq \boxslash \square P$.

Question: When is that an equality?

Commutativity of $\Box$, $\boxslash$

As we saw (without proof),

Lemma. $\Box \boxslash P \leq \boxslash \Box P$.

Theorem. *If* Γ *preserves non-empty intersections, then* $\Box \boxslash P = \boxslash \Box P$.

Commutativity of $\square$, $\boxslash$

As we saw (without proof),

Lemma. $\square \boxslash P \leq \boxslash \square P$.

Theorem. *If* Γ *preserves non-empty intersections, then* $\square \boxslash P = \boxslash \square P$.

In this case, subcoalgebras are closed under arbitrary intersections.

A counterexample

Consider the functor $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ taking a set X to the filters on X.

$\mathcal{F}$ does not preserve non-empty intersections.

A counterexample

Consider the functor $\mathcal{F}:\mathbf{Set}\rightarrow\mathbf{Set}$ taking a set X to the filters on X.

A topological space X may be considered as a $\mathcal{F}$-coalgebra, via the structure map $X\rightarrow\mathcal{F}X$ taking an element $x \in X$ to the neighborhood filter containing x.

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A topological space X may be considered as a $\mathcal{F}$-coalgebra, via the structure map $X\to\mathcal{F}X$ taking an element $x \in X$ to the neighborhood filter containing x.

We will show an example of a space X together with a "coequation" $P \subseteq X$ such that $\Box \boxslash P \neq \boxslash \Box P$, i.e., $\Box \boxslash P \leq \boxslash \Box P$.

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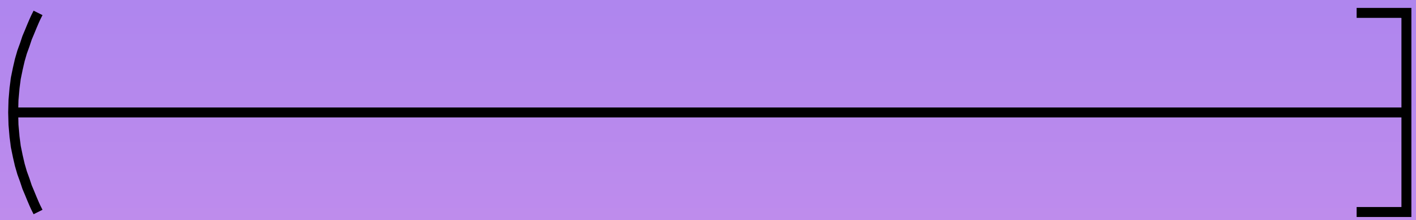

Consider the real interval $(0, 1]$, topologized with open sets of the form $(x, 1]$ for $x \in X$.

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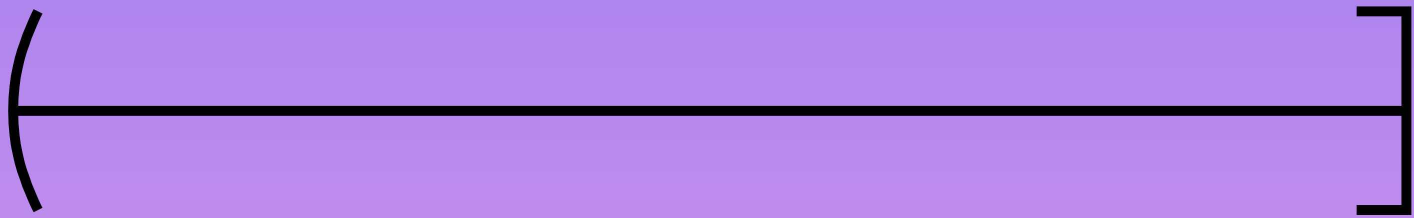

Consider the real interval $(0, 1]$, topologized with open sets of the form $(x, 1]$ for $x \in X$. It is not difficult to show that the only non-trivial endo-invariant subset of $(0, 1]$ is $\{1\}$.

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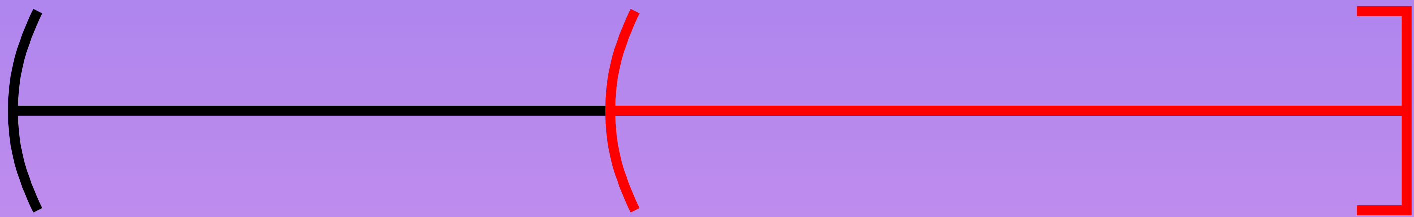

Consider the real interval $(0, 1]$, topologized with open sets of the form $(x, 1]$ for $x \in X$. It is not difficult to show that the only non-trivial endo-invariant subset of $(0, 1]$ is $\{1\}$. Let $P = (\frac{1}{2}, 1]$.

A counterexample

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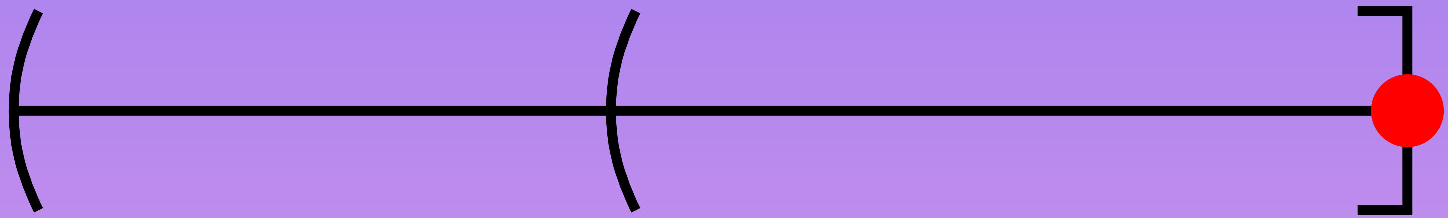
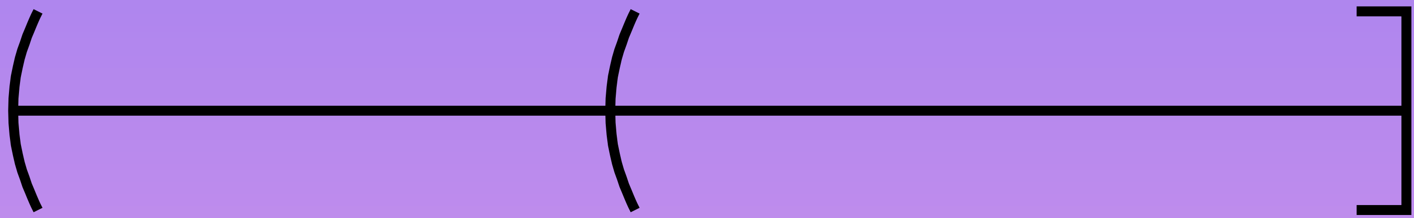


Let $P = (\frac{1}{2}, 1]$. Then $\Box P = P$ and so $\boxslash \Box P = \{1\}$.

A counterexample

We will show an example of a space X together with a "coequation" $P \subseteq X$ such that $\Box \boxslash P \neq \boxslash \Box P$, i.e., $\Box \boxslash P \leq \boxslash \Box P$.



Let $P = (\frac{1}{2}, 1]$. Then $\Box P = P$ and so $\boxslash \Box P = \{1\}$. On the other hand, $\boxslash P = \{1\}$, and so $\Box \boxslash P = \emptyset$.

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