The Formal Dual of Birkhoff’s Completeness Theorem

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University of Nijmegen
Outline

I. Coequations
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II. Conditional coequations
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III. Horn coequations
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IV. Some co-Birkhoff type theorems (again)
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V. Birkhoff’s completeness theorem

VI. Dualizing deductive closure
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VII. The □ operator
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VIII. The ♠ operator

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IX. The invariance theorem
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X. Commutativity of □, □
Coequations

Let $U \dashv H$ and $C \in \mathcal{C}$ be injective with respect to $\mathcal{S}$-morphisms.
A coequation over $C$ is an $\mathcal{S}$-morphism $P \triangleright UHC$ in $\mathcal{C}$. 
Coequations

Let $U \rightarrowtail H$ and $C \in \mathcal{C}$ be injective with respect to $\mathcal{S}$-morphisms.

A coequation over $\mathcal{C}$ is an $\mathcal{S}$-morphism $P \rightarrowtail UHC$ in $\mathcal{C}$. We say $\langle A, \alpha \rangle \models_{\mathcal{C}} P$ just in case for every homomorphism $p : \langle A, \alpha \rangle \rightarrow HC$, we have $\text{Im}(p) \leq P$.

\[
\begin{array}{c}
A \xrightarrow{\forall p} UHC \\
\exists P
\end{array}
\]
Coequations

Let $U \rightarrow H$ and $C \in C$ be injective with respect to $S$-morphisms.

A coequation over $C$ is an $S$-morphism $P \rightarrow UHC$ in $C$.

We say $\langle A, \alpha \rangle \models_{C} P$ just in case for every homomorphism $p: \langle A, \alpha \rangle \rightarrow HC$, we have $\text{Im}(p) \leq P$.

Here, $[P]$ is the largest subcoalgebra of $HC$ contained in $P$. 
Coequations

Let $U \rightharpoonup H$ and $C \in \mathcal{C}$ be injective with respect to $S$-morphisms.

A coequation over $C$ is an $S$-morphism $P \rightharpoonup UHC$ in $\mathcal{C}$.

We say $\langle A, \alpha \rangle \models_{\mathcal{C}} P$ just in case for every homomorphism $p: \langle A, \alpha \rangle \rightarrow HC$, we have $\text{Im}(p) \subseteq P$.

Thus, $\langle A, \alpha \rangle \models_{\mathcal{C}} P$ iff $\langle A, \alpha \rangle \in \text{Proj}([P])$, i.e.,

\[ \text{Hom}(\langle A, \alpha \rangle, HC) \cong \text{Hom}(\langle A, \alpha \rangle, [P]). \]
Example

The cofree coalgebra $H2$
Example

A coequation.
Example

This coalgebra satisfies $P$. 
Example

Under any coloring, the elements of the coalgebra map to elements of $P$. 

The Formal Dual of Birkhoff’s Completeness Theorem – p.4/26
Example

This coalgebra doesn’t satisfy $P$. 
Example

If we paint the circle red, it isn’t mapped to an element of $P$. 
Comparing coequations and equations

<table>
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<tr>
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## Comparing coequations and equations

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$E \xrightarrow{\cong} UF \times \quad P \xrightarrow{\cong} UHC$

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## Comparing coequations and equations

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Comparing coequations and equations

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<td>$\models$ as $i$-projective</td>
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The Formal Dual of Birkhoff's Completeness Theorem – p.5/26
Conditional coequations

Let $P, Q \leq UHC$.

We write $\langle A, \alpha \rangle \models^c P \Rightarrow Q$ just in case, for every $p : \langle A, \alpha \rangle \hookrightarrow HC$ such that $\text{Im}(p) \leq P$, we have $\text{Im}(p) \leq Q$. 
Conditional coequations

Let $P, Q \leq UHC$. We write $\langle A, \alpha \rangle \models_c P \Rightarrow Q$ just in case, for every $p : \langle A, \alpha \rangle \rightarrow HC$ such that $\text{Im}(p) \leq P$, we have $\text{Im}(p) \leq Q$.

\[
\begin{array}{c}
A \xrightarrow{p} UHC \\
\exists \quad P
\end{array}
\quad \Rightarrow 
\quad 
\begin{array}{c}
A \xrightarrow{p} UHC \\
\exists \quad Q
\end{array}
\]
Conditional coequations

Let \( P, Q \leq UHC \).
We write \( \langle A, \alpha \rangle \models_c P \Rightarrow Q \) just in case, for every \( p: \langle A, \alpha \rangle \to HC \) such that \( \text{Im}(p) \leq P \), we have \( \text{Im}(p) \leq Q \).

\[
\langle A, \alpha \rangle \models P \Rightarrow Q \text{ just in case every homomorphism } \langle A, \alpha \rangle \to [P] \text{ factors through } [Q], \text{ i.e., }
\]

\[
\text{Hom}(\langle A, \alpha \rangle, [P]) \cong \text{Hom}(\langle A, \alpha \rangle, [Q]).
\]
Example

Recall our coequation $P$. 
Let $Q$ be the coequation above.
And consider the “conditional coequation” $P \Rightarrow Q$. 
Example

This coalgebra satisfies $P \Rightarrow Q$. 
Example

However we paint it so that it factors through $P$, it also factors through $Q$. 
Example

(It also satisfies \( Q \Rightarrow P \).)
Dualizing negations

Let $P \leq UHC$.
We write $\langle A, \alpha \rangle \models_C \overline{P}$ just in case for every $p : A \rightarrow C$, it is not the case $\text{Im}(\overline{p}) \leq P$. 
Dualizing negations

Let $P \leq UHC$. We write $\langle A, \alpha \rangle \models_c \overline{P}$ just in case for every $p : A \to C$, it is not the case $\text{Im}(\overline{p}) \leq P$.

Equivalently, there is no homomorphism $\langle A, \alpha \rangle \to [P]$, i.e.,

$$\text{Hom}(\langle A, \alpha \rangle, [P]) = \emptyset.$$
Dualizing negations

Let $P \leq UHC$.

We write $\langle A, \alpha \rangle \models_C \overline{P}$ just in case for every $p : A \rightarrow C$, it is not the case $\text{Im}(\overline{p}) \leq P$.

Equivalently, there is no homomorphism $\langle A, \alpha \rangle \rightarrow [P]$, i.e.,

$$\text{Hom}(\langle A, \alpha \rangle, [P]) = \emptyset.$$ 

No matter how we paint $A$, there is some element $a \in A$ that doesn’t land in $P$. 

The Formal Dual of Birkhoff’s Completeness Theorem – p.8/26
Dualizing negations

Let $P \leq UHC$.
We write $\langle A, \alpha \rangle \models_C \overline{P}$ just in case for every $p : A \to C$, it is not the case $\text{Im}(\overline{p}) \leq P$.

No matter how we paint $A$, there is some element $a \in A$ that doesn’t land in $P$.

Note: This does not mean that $\langle A, \alpha \rangle \models \neg P$! “Something in $A$ does not land in $P$,” is not the same as, “Everything in $A$ does not land in $P.””
Example

The coalgebra on the left satisfies $\overline{P}$.
Example

No matter how we paint it, the square does not land in \( P \)
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Some co-Birkhoff-type theorems

Define

\[ \text{Th} \mathbf{V} = \{ P \quad \iff \quad UHC \mid \mathbf{V} \models_c P \} \]
Some co-Birkhoff-type theorems

Define

$$\text{Th } V = \{ P \dashv \vdash UHC \mid V \models_c P \}$$

$$\text{Imp } V = \{ P \Rightarrow_c Q \mid V \models_c P \Rightarrow Q \}$$
Some co-Birkhoff-type theorems

Define

$$\text{Th } V = \{ P \dashv \vdash UHC \mid V \models_c P \}$$

$$\text{Imp } V = \{ P \Rightarrow^C Q \mid V \models_c P \Rightarrow Q \}$$

$$\text{Horn } V = \text{Imp } V \cup \{ \overline{P}^C \mid V \models_c \overline{P} \}$$
Some co-Birkhoff-type theorems

Define

\[ \text{Th} \ V = \{ P \iff UHC \mid V \models_C P \} \]
\[ \text{Imp} \ V = \{ P \Rightarrow^C Q \mid V \models_C P \Rightarrow Q \} \]
\[ \text{Horn} \ V = \text{Imp} \ V \cup \{ \overline{P}^C \mid V \models_C \overline{P} \} \]

Further, let Mod \ S denote the models of \ S for \ S a class of coequations, conditional coequations or Horn coequations.
Some co-Birkhoff-type theorems

Theorem (Birkhoff covariety theorem).

\[ \text{Mod Th} V = \mathcal{SH} \Sigma V \]

Theorem (Quasi-covariety theorem).

\[ \text{Mod Imp} V = \mathcal{H} \Sigma V \]

Theorem (Horn covariety theorem).

\[ \text{Mod Horn} V = \mathcal{H} \Sigma^+ V \]
Birkhoff’s deduction theorem

Fix a set $X$ of variables and let $E$ be a set of equations over $X$. $E$ is deductively closed just in case $E$ satisfies the following:

(i) $x = x \in E$;

(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;

(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;

(iv) $t_1^i = t_2^i \in E$ and $f \in \Sigma \Rightarrow f(t_1^i) = f(t_2^i) \in E$;

(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$. 

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Birkhoff’s deduction theorem

Fix a set $X$ of variables and let $E$ be a set of equations over $X$. $E$ is deductively closed just in case $E$ satisfies the following:

(i) $x = x \in E$;
(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;
(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;
(iv) $t_1^i = t_2^i \in E$ and $f \in \Sigma \Rightarrow f(\vec{t_1}) = f(\vec{t_2}) \in E$;
(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

Items (i) – (iv) ensure that $E$ is a congruence and hence uniquely determines a quotient of $FX$. 
Birkhoff’s deduction theorem

Fix a set $X$ of variables and let $E$ be a set of equations over $X$. $E$ is deductively closed just in case $E$ satisfies the following:

(i) $x = x \in E$;
(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;
(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;
(iv) $t_1^i = t_2^i \in E$ and $f \in \Sigma \Rightarrow f(t_1) = f(t_2) \in E$;
(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

Item (v) ensures that $E$ is a stable $\mathbb{P}$-algebra, i.e., closed under substitutions.
Birkhoff’s deduction theorem

$E$ is deductively closed just in case $E$ satisfies the following:

(i) $x = x \in E$;
(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;
(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;
(iv) $t_1^i = t_2^i \in E$ and $f \in \Sigma \Rightarrow f(t_1^\vec{\cdot}) = f(t_2^\vec{\cdot}) \in E$;
(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

Let $\text{Ded}: \text{Rel}(UFX) \rightarrow \text{Rel}(UFX)$ be the closure operation taking a set $E$ of equations over $X$ to its deductive closure. We can decompose $\text{Ded}$ into two closure operators.
Birkhoff’s deduction theorem

$E$ is deductively closed just in case $E$ satisfies the following:

(i) $x = x \in E$;

(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;

(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;

(iv) $t_1 = t_2 \in E$ and $f \in \Sigma \Rightarrow f(\vec{t}_1) = f(\vec{t}_2) \in E$;

(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

The first takes $E$ to the congruence it generates.
Birkhoff’s deduction theorem

$E$ is deductively closed just in case $E$ satisfies the following:

(i) $x = x \in E$;
(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;
(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;
(iv) $t_1^i = t_2^i \in E$ and $f \in \Sigma \Rightarrow f(t_1^\vec{\cdot}) = f(t_2^\vec{\cdot}) \in E$;
(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

The second closes it under substitution of terms for variables.
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VII. The $\Box$ operator

VIII. The $\blacksquare$ operator

IX. The invariance theorem

X. Commutativity of $\Box$, $\blacksquare$
Dualizing the completeness theorem

Theorem (Birkhoff completeness theorem). For any $E \in \text{Rel}(UFX)$, $\text{Th Mod}(E) = \text{Ded}(E)$
Dualizing the completeness theorem

Theorem (Birkhoff completeness theorem). For any $E \in \text{Rel}(UFX)$, $\text{Th Mod}(E) = \text{Ded}(E)$

Compare this to the variety theorem.

Theorem (Birkhoff variety theorem).

$\text{Mod Th } V = \mathcal{HSP}V$
Dualizing the completeness theorem

Theorem (Birkhoff completeness theorem). For any $E \in \text{Rel}(UFX)$, $\text{Th Mod}(E) = \text{Ded}(E)$

$\text{Th Mod}(E')$ satisfies the following fixed point description.

- $\text{Mod}(E) \models \text{Th Mod}(E)$;
- If $\text{Mod}(E) \models E'$, then $E' \subseteq \text{Th Mod}(E)$. 
Dualizing the completeness theorem

$\text{Th Mod}(E)$ satisfies the following fixed point description.

- $\text{Mod}(E) \models \text{Th Mod}(E)$;
- If $\text{Mod}(E) \models E'$, then $E' \subseteq \text{Th Mod}(E)$.

We dualize this fixed point description to yield its coalgebraic analogue. We call the analogue the “generating coequation for $\text{Mod}(P)$”, written $\text{Gen Mod}(P)$. 

Dualizing the completeness theorem

We dualize this fixed point description to yield its coalgebraic analogue. We call the analogue the “generating coequation for \( \text{Mod}(P) \)”, written \( \text{Gen Mod}(P) \).

\( \text{Gen Mod}(P) \) satisfies the following fixed point description.

- \( \text{Mod}(P) \models \text{Gen Mod}(E) \);
- If \( \text{Mod}(P) \models P' \), then \( \text{Gen Mod}(P) \subseteq P' \).
Dualizing the completeness theorem

We dualize this fixed point description to yield its coalgebraic analogue. We call the analogue the “generating coequation for Mod($P$)”, written Gen Mod($P$).

Gen Mod($P$) satisfies the following fixed point description.

- Mod($P$) ⊨ Gen Mod($E$);
- If Mod($P$) ⊨ $P'$, then Gen Mod($P$) ⊆ $P'$.

Recall that sets of equations correspond to coequations, so this is an appropriate dualization.
Dualizing the completeness theorem

We dualize this fixed point description to yield its coalgebraic analogue. We call the analogue the “generating coequation for $\text{Mod}(P)$”, written $\text{Gen Mod}(P)$.

$\text{Gen Mod}(P)$ satisfies the following fixed point description.

- $\text{Mod}(P) \models \text{Gen Mod}(E)$;
- If $\text{Mod}(P) \models P'$, then $\text{Gen Mod}(P) \subseteq P'$.

Recall that sets of equations correspond to coequations, so this is an appropriate dualization.

A generating coequation gives a measure of the “coequational commitment” of $V$. 

Dualizing deductive closure

**Theorem (Birkhoff completeness theorem).** For any \( E \in \text{Rel}(UFX) \), \( \text{Th Mod}(E) = \text{Ded}(E) \)

To dualize \( \text{Ded} \), we consider again its components.

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<td>( q : FX \rightarrow \langle Q, \nu \rangle )</td>
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Dualizing deductive closure

Theorem (Birkhoff completeness theorem). For any $E \in \text{Rel}(UFX)$, $\text{Th Mod}(E) = \text{Ded}(E)$

To dualize Ded, we consider again its components.

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Dualizing deductive closure

Theorem (Birkhoff completeness theorem). For any $E \in \text{Rel}(UFX)$, $\text{Th Mod}(E) = \text{Ded}(E)$

To dualize Ded, we consider again its components.

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<td>Closure under substitution</td>
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VIII. The ☐ operator
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The modal operator □

Let \( P, Q \vdash A \) be given. We write \( P \vdash Q \) if there is a map \( P \rightarrow Q \) such that the diagram below commutes.

\[
\begin{array}{ccc}
P & \rightarrow & Q \\
\downarrow & & \downarrow \\
A & & \\
\end{array}
\]
The modal operator

Let $P, Q \rightarrow A$ be given. We write $P \vdash Q$ if there is a map $P \rightarrow Q$ such that the diagram below commutes.

\[
\begin{array}{ccc}
P & \xrightarrow{\sim} & Q \\
\downarrow & & \downarrow \\
A & & \\
\end{array}
\]

In fact, $P \rightarrow Q$ is necessarily an $S$-morphism.
The modal operator ☐

Let ☐ : Sub(\(UHC\)) \rightarrow Sub(\(UHC\)) be the composite \(U[\ - \].\nIn other terms, ☐ is a comonad taking a coequation \(P\) to the largest subcoalgebra \(\langle A, \alpha \rangle\) of \(HC\) such that \(A \leq P\).
The modal operator □

Let □ : Sub(UHC) → Sub(UHC) be the composite \( U[\_\_] \). In other terms, □ is a comonad taking a coequation \( P \) to the largest subcoalgebra \( \langle A, \alpha \rangle \) of \( HC \) such that \( A \leq P \).

As is well-known, if \( \Gamma \) preserves pullbacks of \( S \)-morphisms, then □ is an \( S4 \) operator.

(i) If \( P \vdash Q \) then □\( P \vdash □Q \);
(ii) □\( P \vdash P \);
(iii) □\( P \vdash □□P \);
(iv) □(\( P \rightarrow Q \)) \vdash □P \rightarrow □Q ;
The modal operator □

Let □:Sub(\(UHC\)) \to Sub(\(UHC\)) be the composite \(U[-]\). In other terms, □ is a comonad taking a coequation \(P\) to the largest subcoalgebra \(\langle A, \alpha\rangle\) of \(HC\) such that \(A \leq P\).

(i) If \(P \vdash Q\) then \(□P \vdash □Q\);
(ii) \(□P \vdash P\);
(iii) \(□P \vdash □ □ P\);
(iv) \(□(P \to Q) \vdash □P \to □Q\);

(i) follows from functoriality.
The modal operator □

Let □ : Sub(UHC) → Sub(UHC) be the composite U[−]. In other terms, □ is a comonad taking a coequation P to the largest subcoalgebra ⟨A, α⟩ of HC such that A ≤ P.

(i) If P ⊨ Q then □P ⊨ □Q;
(ii) □P ⊨ P;
(iii) □P ⊨ □□P;
(iv) □(P → Q) ⊨ □P → □Q;

(ii) and (iii) are the counit and comultiplication of the comonad.
The modal operator □

Let □ : Sub(UHC) → Sub(UHC) be the composite U[−]. In other terms, □ is a comonad taking a coequation P to the largest subcoalgebra ⟨A, α⟩ of HC such that A ≤ P.

(i) If P ⊩ Q then □P ⊩ □Q;
(ii) □P ⊩ P;
(iii) □P ⊩ □□P;
(iv) □(P → Q) ⊩ □P → □Q;

(iv) follows from the fact that U : EΓ → E preserves finite meets.
The modal operator □

(i) If $P \vdash Q$ then $\square P \vdash \square Q$;
(ii) $\square P \vdash P$;
(iii) $\square P \vdash \square \square P$;
(iv) $\square (P \rightarrow Q) \vdash \square P \rightarrow \square Q$;

Proof.

\[
\frac{P \rightarrow Q \vdash P \rightarrow Q}{\square (P \rightarrow Q) \wedge P \vdash Q}
\]

By the counit of adjunction $- \wedge P \vdash P \rightarrow -$.
The modal operator □

(i) If \( P \vdash Q \) then \( \Box P \vdash \Box Q \);
(ii) \( \Box P \vdash P \);
(iii) \( \Box P \vdash \Box \Box P \);
(iv) \( \Box (P \rightarrow Q) \vdash \Box P \rightarrow \Box Q \);

Proof.

\[
\begin{align*}
(P \rightarrow Q) \land P & \vdash Q \\
\Box((P \rightarrow Q) \land P) & \vdash \Box Q
\end{align*}
\]

By (i).
The modal operator □

(i) If $P \vdash Q$ then $\square P \vdash \square Q$;
(ii) $\square P \vdash P$;
(iii) $\square P \vdash \square \square P$;
(iv) $\square (P \rightarrow Q) \vdash \square P \rightarrow \square Q$;

Proof.

\[
\square((P \rightarrow Q) \land P) \vdash \square Q \\
\square(P \rightarrow Q) \land \square P \vdash \square Q
\]

Because $\square$ preserves meets.
The modal operator □

(i) If $P \vdash Q$ then $\square P \vdash \square Q$;
(ii) $\square P \vdash P$;
(iii) $\square P \vdash \square \square P$;
(iv) $\square (P \rightarrow Q) \vdash \square P \rightarrow \square Q$;

Proof.

\[
\begin{align*}
\square (P \rightarrow Q) \land \square P & \vdash \square Q \\
\hline
\square (P \rightarrow Q) & \vdash \square P \rightarrow \square Q
\end{align*}
\]

Again, by the adjunction $\neg \land P \vdash P \rightarrow \neg$. □
Invariant coequations

Let \( f: \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle \) and \( P \xrightarrow{} A \) be given. We let \( \exists_f P \) denote the image of the composite \( P \xrightarrow{} A \xrightarrow{} B \).

\[
\begin{align*}
P & \xrightarrow{} \exists_f P \\
A & \xrightarrow{} B
\end{align*}
\]
Invariant coequations

Let $P \subseteq UHC$. We say that $P$ is endomorphism-invariant just in case, for every “repainting”

$$p: UHC \rightarrow C,$$

equivalently, every homomorphism $\tilde{p}: HC \rightarrow HC$, we have

$$\exists \tilde{p} P \leq P.$$
Invariant coequations

Let $P \subseteq UHC$. We say that $P$ is endomorphism-invariant just in case, for every “repainting”

$$p: UHC \longrightarrow C,$$

equivalently, every homomorphism $\tilde{p}: HC \rightarrow HC$, we have

$$\exists c \in UHC (\tilde{p}(c) = x \land P(c)) \vdash P(x).$$
Invariant coequations

Let $P \subseteq UHC$. We say that $P$ is endomorphism-invariant just in case, for every “repainting”

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equivalently, every homomorphism $\tilde{p}: HC \rightarrow HC$, we have

$$\exists c \in UHC (\tilde{p}(c) = x \land P(c)) \vdash P(x).$$

In other words, however we repaint $HC$, the elements of $P$ are again (under this new coloring) elements of $P$. 
Definition of \[ \mathcal{I}_P \]

Let \( P \subseteq UHC \). Define

\[
\mathcal{I}_P = \{ Q \leq UHC \mid \forall p: HC \rightarrow HC (\exists_p Q \leq P) \}.
\]
Definition of $\mathcal{I}$

Let $P \subseteq UHC$. Define

$$
\mathcal{I}_P = \{ Q \leq UHC \mid \forall p: HC \rightarrow HC (\exists_p Q \leq P) \}.
$$

That is, $\mathcal{I}_P$ is the collection of all those coequations $Q$ such that, however we “repaint” $UHC$, the image of $Q$ still lands in $P$. 

Definition of \(\mathcal{I}_P\)

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\[
\mathcal{I}_P = \{ Q \leq UHC \mid \forall p : HC \rightarrow HC (\exists_p Q \leq P) \}.
\]

That is, \(\mathcal{I}_P\) is the collection of all those coequations \(Q\) such that, however we “repaint” \(UHC\), the image of \(Q\) still lands in \(P\).

In particular, if \(Q \in \mathcal{I}_P\), then \(Q \vdash P\).
Definition of $\blacksquare$

Let $P \subseteq UHC$. Define

$$\mathcal{I}_P = \{ Q \leq UHC \mid \forall p : HC \rightarrow HC (\exists_p Q \leq P) \}.$$ 

We define a functor $\blacksquare : \text{Sub}(UHC) \rightarrow \text{Sub}(UHC)$ by

$$\blacksquare P = \bigvee \mathcal{I}_P.$$ 

Then $\blacksquare P$ is the greatest invariant subobject of $UHC$ contained in $P$. 

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Definition of $\Box$

We define a functor $\Box : \text{Sub}(UHC) \to \text{Sub}(UHC)$ by

$$\Box P = \bigvee \mathcal{I}_P.$$ 

That is, $\Box P$ satisfies the following:

- For all $p : HC \to HC$, $\exists p \Box P \vdash \Box P$. 

Definition of \(\mathcal{D}\)

We define a functor \(\mathcal{D} : \text{Sub}(UHC) \to \text{Sub}(UHC)\) by

\[
\mathcal{D} P = \sqrt{I_P}. 
\]

That is, \(\mathcal{D} P\) satisfies the following:

- For all \(p : HC \to HC\), \(\exists p \mathcal{D} P \vdash \mathcal{D} P\).
- If \(Q \vdash P\) and for all \(p : HC \to HC\), \(\exists p Q \vdash Q\), then \(Q \vdash \mathcal{D} P\).
Example (cont.)

The coequation $P$. 
Example (cont.)

\[ P \text{ is not invariant.} \]
Example (cont.)

The coequation \( \square P \).
is S4

One can show that □ is an S4 operator.

(i) If $P \vdash Q$ then $\square P \vdash \square Q$;
(ii) $\square P \vdash P$;
(iii) $\square P \vdash \square \square P$;
(iv) $\square (P \rightarrow Q) \vdash \square P \rightarrow \square Q$;
☐ is S4

One can show that ☐ is an S4 operator.

(i) If $P \vdash Q$ then $\Box P \vdash \Box Q$;

(ii) $\Box P \vdash P$;

(iii) $\Box P \vdash \Box \Box P$;

(iv) $\Box (P \rightarrow Q) \vdash \Box P \rightarrow \Box Q$;

(i) - (iii) follow from the fact that ☐ is a comonad, as before.
is S4

One can show that □ is an S4 operator.

(i) If $P \vdash Q$ then $\Box P \vdash \Box Q$;
(ii) $\Box P \vdash P$;
(iii) $\Box P \vdash \Box \Box P$;
(iv) $\Box (P \rightarrow Q) \vdash \Box P \rightarrow \Box Q$;

(iv) requires an argument that the meet of two invariant co-equations is again invariant. This is not difficult.
Outline

I. Coequations
II. Conditional coequations
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V. Birkhoff’s completeness theorem
VI. Dualizing deductive closure
VII. The $\Box$ operator
VIII. The $\blacksquare$ operator
IX. The invariance theorem
X. Commutativity of $\Box$, $\blacksquare$
The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ iff $\langle A, \alpha \rangle \models \Box P$. 
The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ iff $\langle A, \alpha \rangle \models \Box P$.

Lemma. $\langle A, \alpha \rangle \models P$ iff $\langle A, \alpha \rangle \models \Box P$. 
The invariance theorem

Lemma. \( \langle A, \alpha \rangle \models P \) iff \( \langle A, \alpha \rangle \models \Box \Box P \).

Lemma. \( [\Box P] \models P \).
The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ iff $\langle A, \alpha \rangle \models \Box \Box P$.
Lemma. $[\Box P] \models P$.
Lemma. $\Box \Box P \leq \Box \Box P$, i.e., if $P$ is invariant, then so is $\Box P$. 
The invariance theorem

Lemma. \( \langle A, \alpha \rangle \models P \) iff \( \langle A, \alpha \rangle \models \square \square P \).

Lemma. \( [\square P] \models P \).

Lemma. \( \square \square P \leq \square \square P \).

Theorem. Gen Mod \( P = \square \square P \).
The invariance theorem

**Lemma.** \( \langle A, \alpha \rangle \models P \iff \langle A, \alpha \rangle \models \Box \Box P \).

**Lemma.** \([\Box P] \models P \).

**Lemma.** \( \Box \Box P \leq \Box \Box P \).

**Theorem.** Gen Mod \( P = \Box \Box P \).

**Proof.** From the above, we see that Mod \( P \models \Box \Box P \).
The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ iff $\langle A, \alpha \rangle \models \Box \Box P$.

Lemma. $[\Box P] \models P$.

Lemma. $\Box \Box P \leq \Box \Box P$.

Theorem. Gen Mod $P = \Box \Box P$.

Proof. From the above, we see that Mod $P \models \Box \Box P$.

Suppose that Mod $P \models Q$. Then $[\Box P] \models Q$.  

$$
\Box
$$

The Formal Dual of Birkhoff's Completeness Theorem – p.23/26
The invariance theorem

Lemma. \( \langle A, \alpha \rangle \models P \iff \langle A, \alpha \rangle \models \Box \Box P. \)

Lemma. \([\Box P] \models P.\]

Lemma. \(\Box \Box P \leq \Box \Box P.\)

Theorem. Gen Mod \(P = \Box \Box P.\)

Proof. From the above, we see that Mod \(P \models \Box \Box P.\)
Suppose that Mod \(P \models Q.\) Then \([\Box P] \models Q.\) Hence:

\[
U[\Box P] \iff UHC
\]

The Formal Dual of Birkhoff’s Completeness Theorem – p.23/26
The invariance theorem

Lemma. $\langle A, \alpha \rangle \models P$ iff $\langle A, \alpha \rangle \models \Box \Box P$.

Lemma. $[\Box P] \models P$.

Lemma. $\Box \Box P \leq \Box \Box P$.

Theorem. Gen Mod $P = \Box \Box P$.

Proof. From the above, we see that Mod $P \models \Box \Box P$.

Suppose that Mod $P \models Q$. Then $[\Box P] \models Q$. Hence:

That is, $\Box \Box P \vdash Q$. □
Commutativity of $\Box, \lhd$

As we saw (without proof),

**Lemma.** $\Box \lhd P \leq \lhd \Box P$.

That is, the greatest subcoalgebra of an endomorphism invariant predicate is itself invariant.
Commutativity of $\square, \mathcal{F}$

As we saw (without proof),

**Lemma.** $\square \mathcal{F} P \leq \mathcal{F} \square P$.

**Question:** When is that an equality?
Commutativity of □, □

As we saw (without proof),

**Lemma.** □ □ P ≤ □ □ P.

**Theorem.** If $\Gamma$ preserves non-empty intersections, then □ □ P = □ □ P.
Commutativity of $\Box$, $\Diamond$

As we saw (without proof),

**Lemma.** $\Box \Diamond P \leq \Box \Box P$.

**Theorem.** If $\Gamma$ preserves non-empty intersections, then $\Box \Diamond P = \Diamond \Box P$.

In this case, subcoalgebras are closed under arbitrary intersections.
A counterexample

Consider the functor $\mathcal{F} : \text{Set} \rightarrow \text{Set}$ taking a set $X$ to the filters on $X$.

$\mathcal{F}$ does not preserve non-empty intersections.
A counterexample

Consider the functor $\mathcal{F}: \text{Set} \to \text{Set}$ taking a set $X$ to the filters on $X$.

A topological space $X$ may be considered as a $\mathcal{F}$-coalgebra, via the structure map $X \to \mathcal{F}X$ taking an element $x \in X$ to the neighborhood filter containing $x$. 
A counterexample

Consider the functor $\mathcal{F}: \text{Set} \to \text{Set}$ taking a set $X$ to the filters on $X$.

A topological space $X$ may be considered as a $\mathcal{F}$-coalgebra, via the structure map $X \to \mathcal{F}X$ taking an element $x \in X$ to the neighborhood filter containing $x$.

We will show an example of a space $X$ together with a “coequation” $P \subseteq X$ such that $\Box \Box P \neq \Box \Box P$, i.e., $\Box \Box P \leq \Box \Box P$. 

The Formal Dual of Birkhoff's Completeness Theorem – p.25/26
A counterexample

We will show an example of a space $X$ together with a “coequation” $P \subseteq X$ such that $\square \square P \neq \square \square P$, i.e., $\square \square P \leq \square \square P$.

Consider the real interval $(0, 1]$, topologized with open sets of the form $(x, 1]$ for $x \in X$. 

The Formal Dual of Birkhoff's Completeness Theorem – p.25/26
A counterexample

We will show an example of a space $X$ together with a “coequation” $P \subseteq X$ such that $\Box \Box P \neq \Box \Box P$, i.e., $\Box \Box P \leq \Box \Box P$.

Consider the real interval $(0, 1]$, topologized with open sets of the form $(x, 1]$ for $x \in X$. It is not difficult to show that the only non-trivial endo-invariant subset of $(0, 1]$ is $\{1\}$. 
A counterexample

We will show an example of a space $X$ together with a “coequation” $P \subseteq X$ such that $\square \square P \neq \square \square P$, i.e., $\square \square P \leq \square \square P$.

Consider the real interval $(0, 1]$, topologized with open sets of the form $(x, 1]$ for $x \in X$. It is not difficult to show that the only non-trivial endo-invariant subset of $(0, 1]$ is $\{1\}$. Let $P = (\frac{1}{2}, 1]$. 
A counterexample

We will show an example of a space $X$ together with a “coequation” $P \subseteq X$ such that $\Box \Box P \neq \Box \Box P$, i.e., $\Box \Box P \leq \Box \Box P$.

Let $P = (\frac{1}{2}, 1]$. Then $\Box P = P$ and so $\Box \Box P = \{1\}$. 
A counterexample

We will show an example of a space $X$ together with a “coequation” $P \subseteq X$ such that $\square \Box P \neq \Box \square P$, i.e., $\square \Box P \leq \Box \square P$.

Let $P = (\frac{1}{2}, 1]$. Then $\square P = P$ and so $\Box \square P = \{1\}$. On the other hand, $\square P = \{1\}$, and so $\Box \square P = \emptyset$. 
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