Modal Operators for Coequations

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Outline

I. The co-Birkhoff Theorem

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II. Deductive completeness

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III. The $\Box$ operator

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IV. The $\boxtimes$ operator

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II. Deductive completeness
III. The $\Box$ operator
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V. The invariance theorem

The Birkhoff variety theorem

Let $\mathbb{P}:\mathbf{Set}\to\mathbf{Set}$ be a polynomial functor, and X an infinite set of variables.

Theorem (Birkhoff's variety theorem (1935)). *A full subcategory* $\mathbf{V}$ *of* $\mathbf{Set}^{\mathbb{P}}$ *is closed under*

- *products,*
- *subalgebras and*
- *quotients (codomains of regular epis)*

just in case $\mathbf{V}$ *is definable by a set of equations* E *over* X*, i.e.,*

$$\mathbf{V} = \{\langle A, \alpha\rangle \mid \langle A, \alpha\rangle \models E\}.$$

The covariety theorem

Let $\Gamma : \mathcal{E} \to \mathcal{E}$ be a functor bounded by $C \in \mathcal{E}$.

Theorem. *A full subcategory* $\mathbf{V}$ *of* $\mathcal{E}_\Gamma$ *is closed under*

- *coproducts,*
- *images (codomains of epis) and*
- *(regular) subcoalgebras*

just in case $\mathbf{V}$ *is definable by a coequation* φ *over* C*, i.e.,*

$$\mathbf{V} = \{\langle A, \alpha\rangle \mid \langle A, \alpha\rangle \models \varphi\}.$$

Coequations

A coequation over C is a subobject of UHC, the cofree coalgebra over C.

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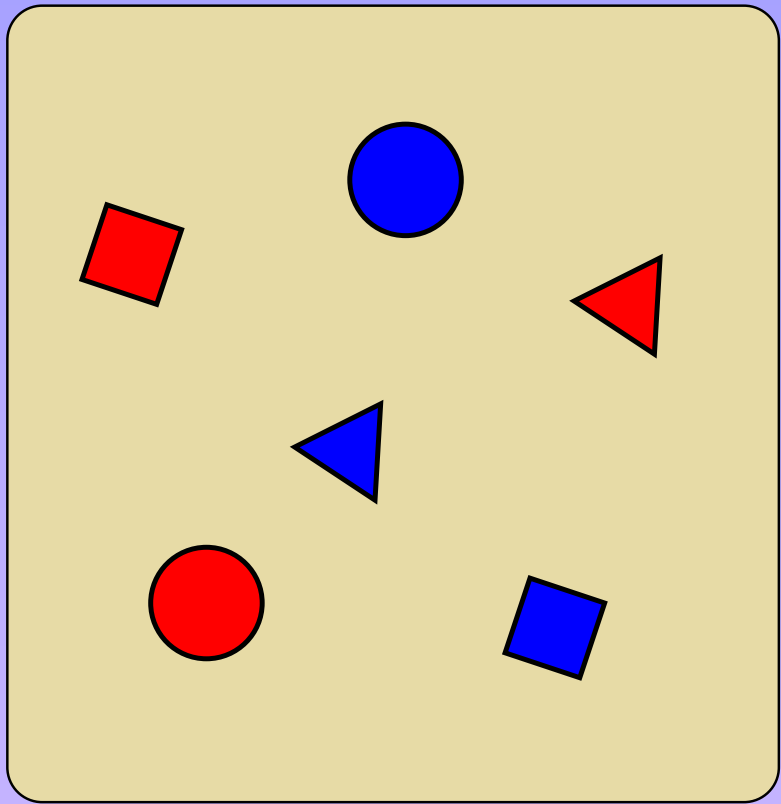
A coalgebra $\langle A,\, \alpha\rangle$ satisfies φ just in case, for every homomorphism

$$p : \langle A,\, \alpha\rangle \longrightarrow HC,$$

the image of p is contained in φ (i.e., $\mathsf{Im}(p) \leq \varphi$).

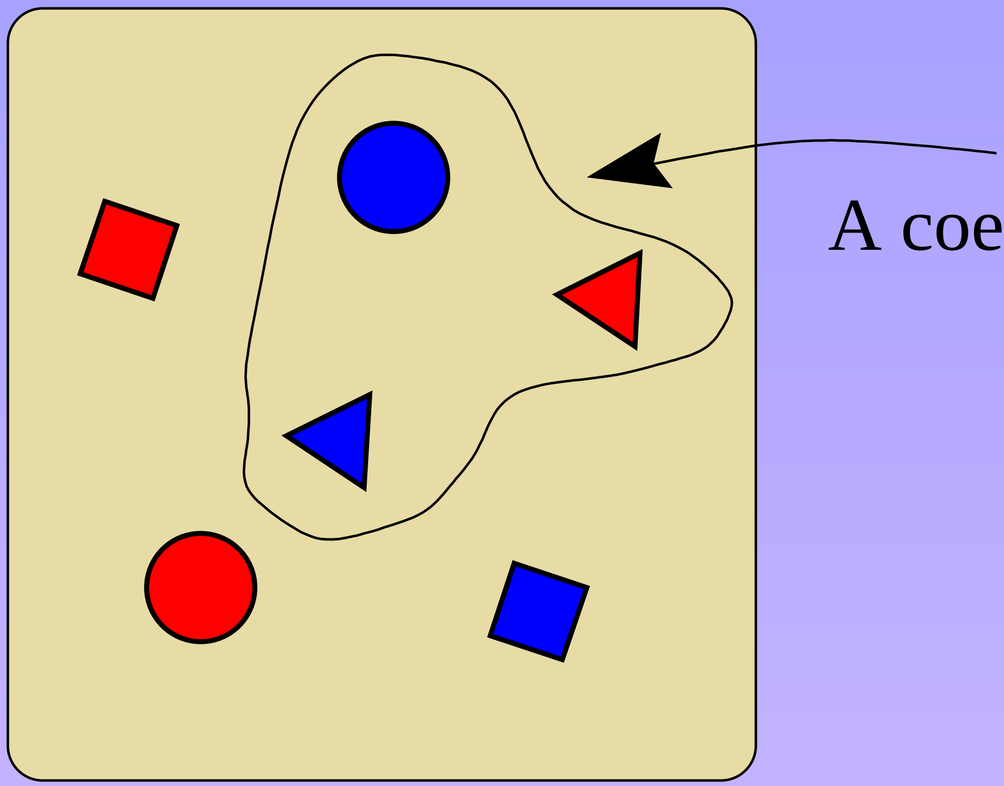
$$\begin{array}{ccc} U\langle A,\, \alpha\rangle & \longrightarrow & UHC \\ & \searrow & \uparrow \\ & & \varphi \end{array}$$

Example



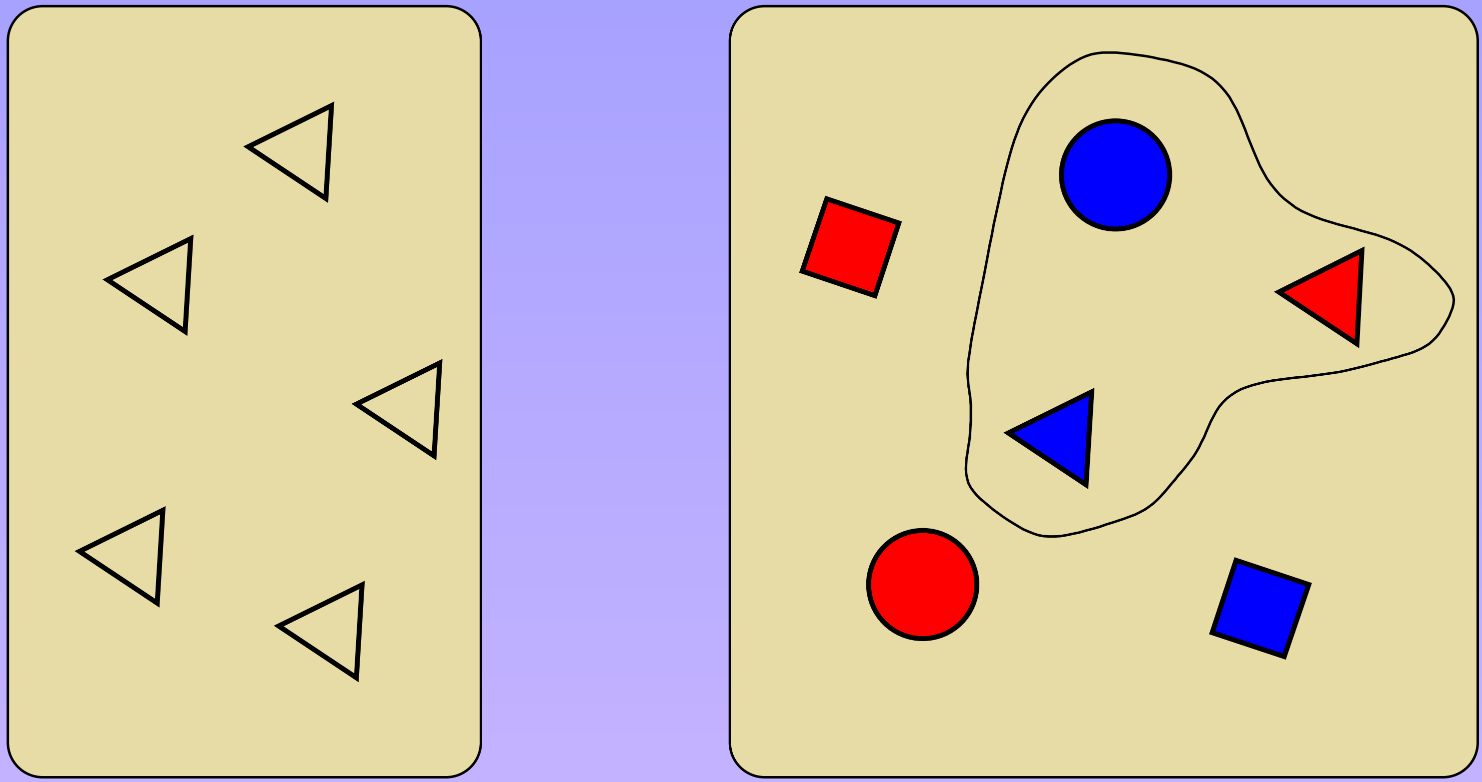
The cofree coalgebra $H2$

Example



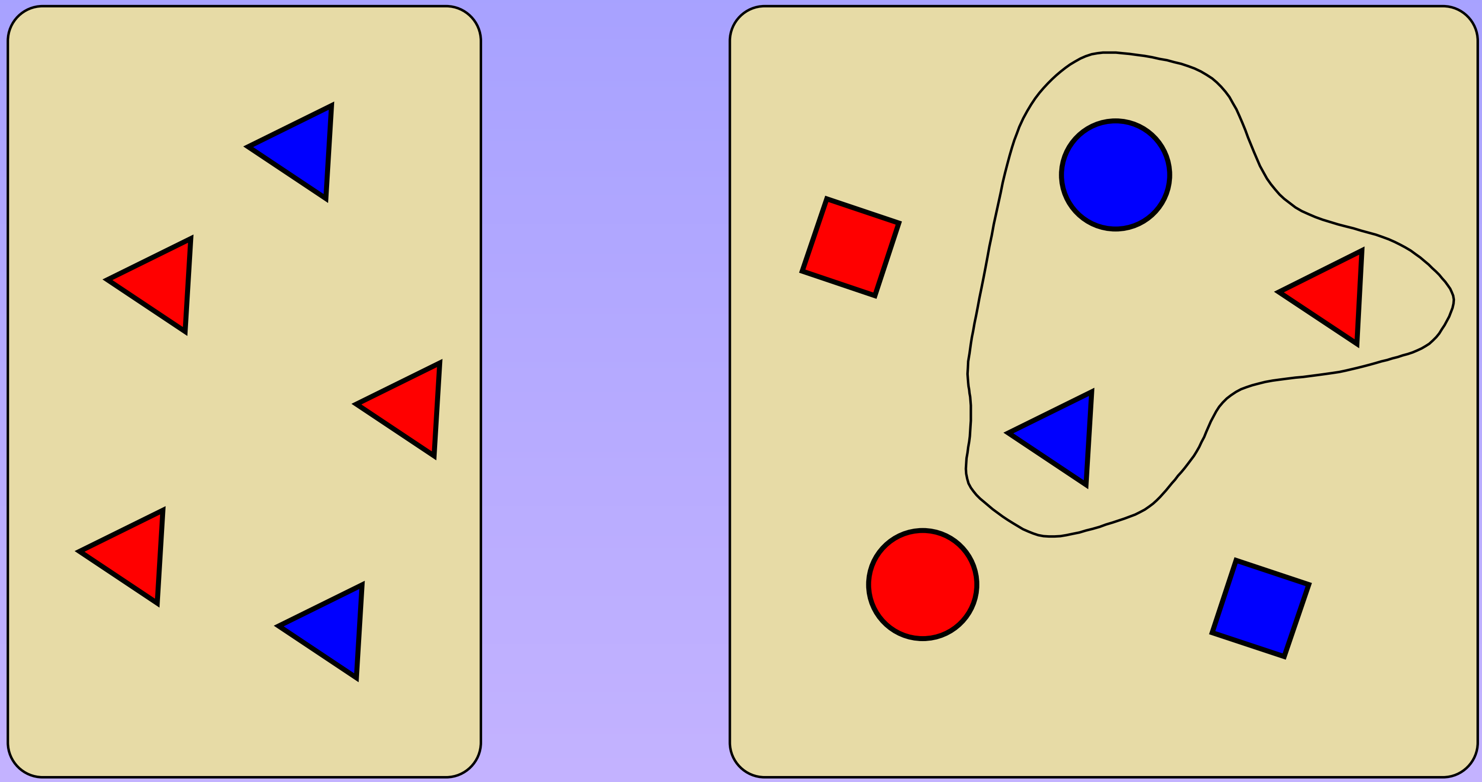
A coequation.

Example



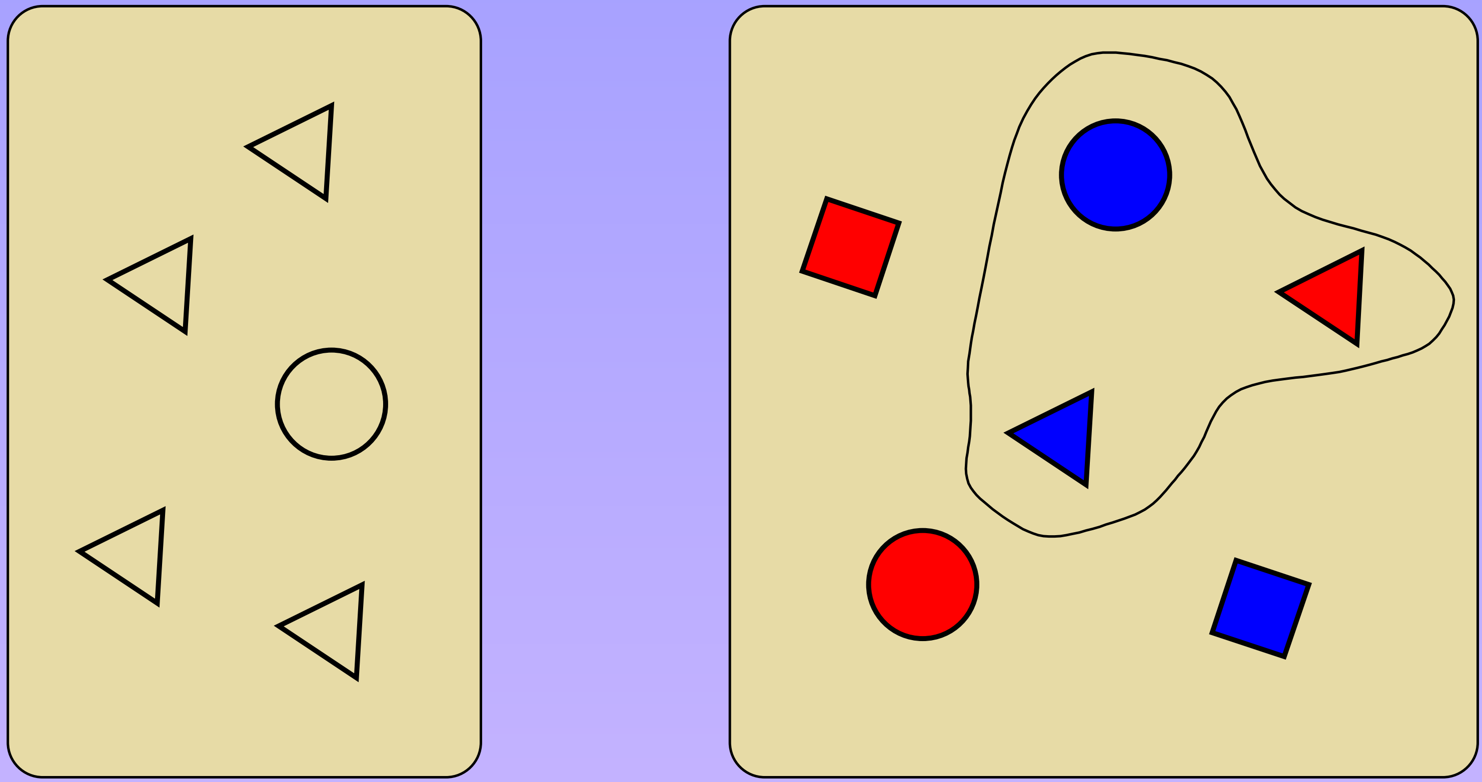
This coalgebra satisfies φ.

Example



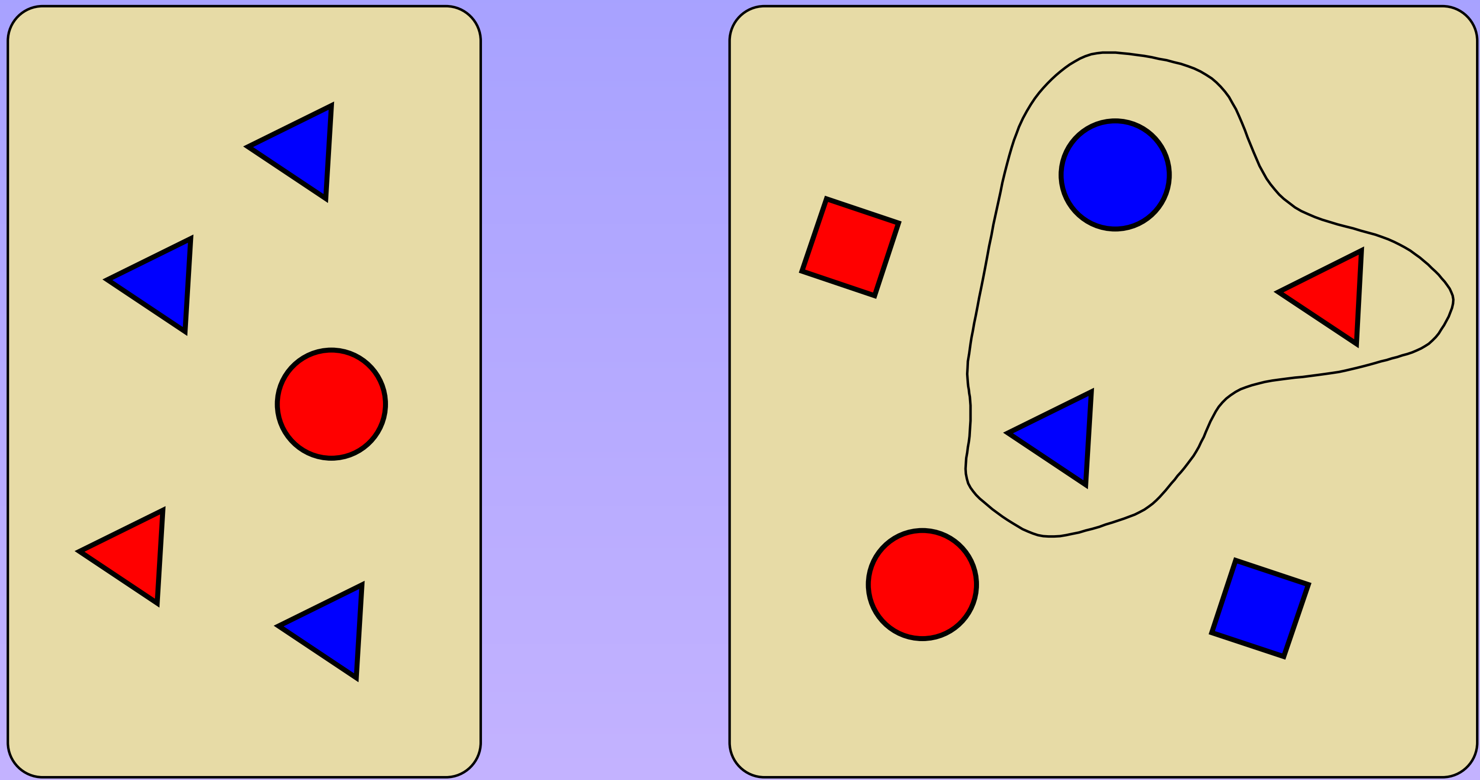
Under any coloring, the elements of the coalgebra map to elements of φ.

Example



This coalgebra doesn't satisfy φ.

Example



If we paint the circle red, it isn't mapped to an element of φ.

Coequations as predicates

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Since a coequation φ over C is just a subobject of UHC, a coequation can be viewed as a predicate over UHC. Hence, the coequations over C come with a natural structure. We can build new coequations out of old via $\wedge$, $\neg$, $\forall$, etc.

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$\langle A,\ \alpha\rangle$ satisfies φ just in case, for every $p : \langle A,\ \alpha\rangle \twoheadrightarrow HC$,

$$\exists_{a \in A}(p(a) = x) \vdash \varphi(x).$$

Birkhoff's deduction theorem

A set of equations E is deductively closed just in case E satisfies the following:

(i) $x = x \in E$;

(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;

(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;

(iv) E is closed under the $\mathbb{P}$-operations;

(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

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Items (i)–(iv) ensure that E is a congruence and hence uniquely determines a quotient of FX.

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Item (v) ensures that E is a stable $\mathbb{P}$-algebra, i.e., closed under substitutions.

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Theorem (Birkhoff completeness theorem). *$E = \mathcal{T}h_{\mathsf{Eq}}(\mathbf{V})$ for some class $\mathbf{V}$ iff E is deductively closed.*

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- Closing E under substitutions corresponds to taking the largest invariant coequation contained in φ.

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- Closing E under substitutions corresponds to taking the largest invariant coequation contained in φ.

Theorem (Invariance theorem). *φ is a generating coequation just in case φ is an invariant subcoalgebra of HC.*

Theories/Generating coequations

A set of equations E is the equational theory for some class $\mathbf{V}$ of algebras iff

- $\mathbf{V} \models E$;
- If $\mathbf{V} \models E'$, then $E' \subseteq E$.

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A generating coequation gives a measure of the "coequational commitment" of $\mathbf{V}$.

Invariant coequations

Let $\varphi \subseteq UHC$. We say that φ is invariant just in case, for every "repainting"

$$p: UHC \longrightarrow C,$$

equivalently, every homomorphism $\widetilde{p}: HC \rightarrow HC$, we have

$$\exists_{\widetilde{p}} \varphi \leq \varphi.$$

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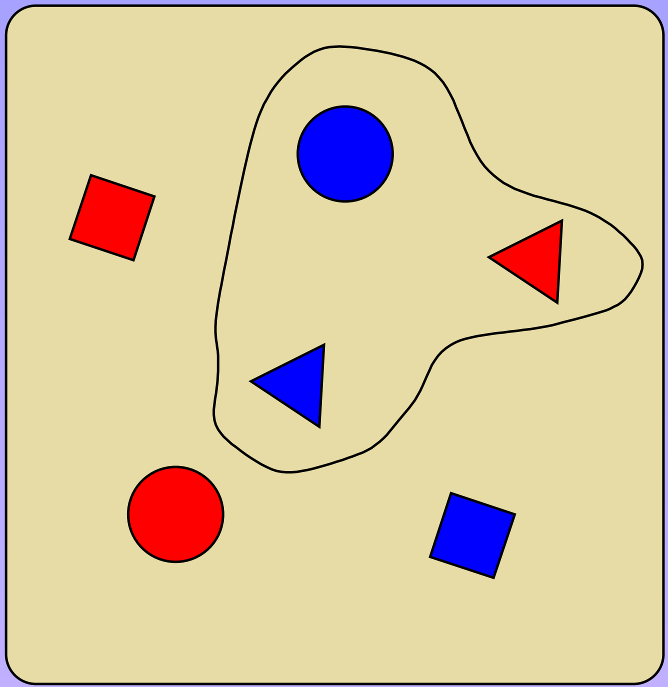
$$\exists_{c \in UHC}(\widetilde{p}(c) = x \wedge \varphi(c)) \vdash \varphi(x).$$

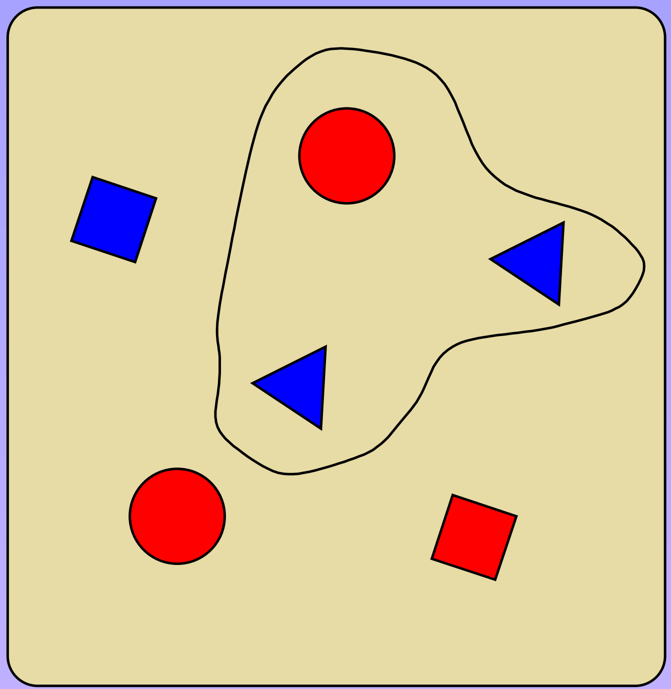
In other words, however we repaint HC, the elements of φ are again (under this new coloring) elements of φ.

Example (cont.)

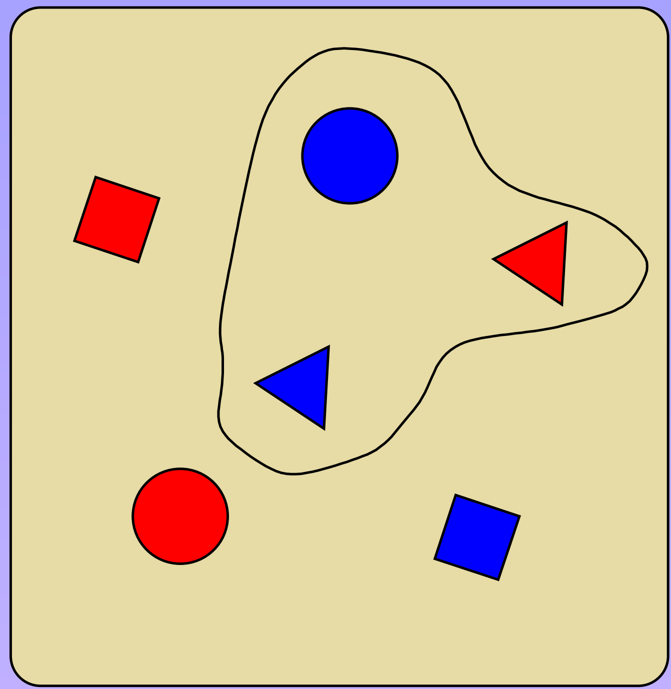


The coequation φ.

Example (cont.)



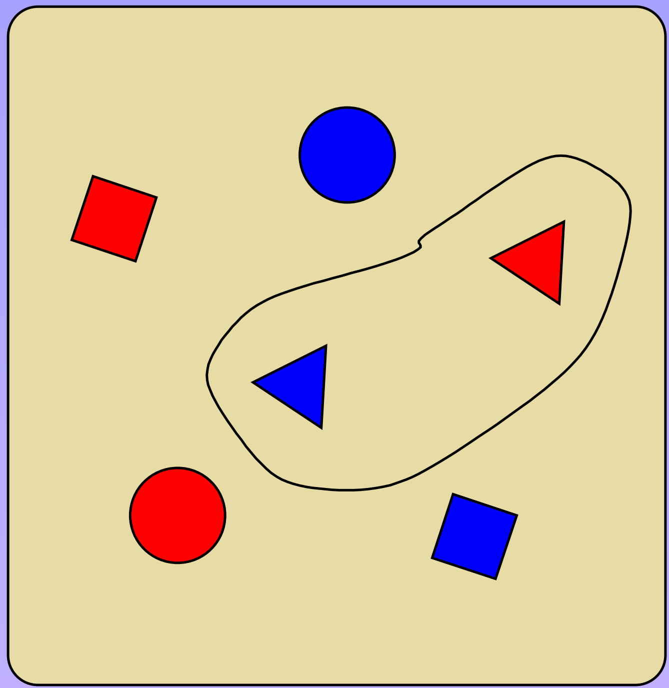
The repainted coalgebra



The cofree coalgebra

φ is not invariant.

Example (cont.)



The coequation $\boxtimes\varphi$.

The modal operator $\Box$

Let $\Box : \mathsf{Sub}(UHC) \to \mathsf{Sub}(UHC)$ be the comonad taking a coequation φ to the largest subcoalgebra $\langle A, \alpha \rangle$ of HC such that $A \leq \varphi$.

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As is well-known, if Γ preserves pullbacks of subobjects, then $\Box$ is an **S4** operator.

(i) If $\varphi \vdash \psi$ then $\Box\varphi \vdash \Box\psi$;

(ii) $\Box\varphi \vdash \varphi$;

(iii) $\Box\varphi \vdash \Box\Box\varphi$;

(iv) $\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi$;

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(i) follows from functoriality.

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(ii) and (iii) are the counit and comultiplication of the comonad.

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(iv) $\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi$;

(iv) follows from the fact that $U : \mathcal{E}_\Gamma \to \mathcal{E}$ preserves finite meets.

Definition of $\boxtimes$

Let $\varphi \subseteq UHC$. Define

$$\mathcal{I}_\varphi = \{\psi \leq UHC \mid \forall p : HC \longrightarrow HC\,(\exists_p \psi \leq \varphi)\}.$$

We define a functor $\boxtimes : \mathsf{Sub}(UHC) \to \mathsf{Sub}(UHC)$ by

$$\boxtimes\varphi = \bigvee \mathcal{I}_\varphi.$$

Then $\boxtimes\varphi$ is the greatest invariant subobject of UHC contained in φ.

$\boxtimes$ is S4

One can show that $\boxtimes$ is an **S4** operator.

(i) If $\varphi \vdash \psi$ then $\boxtimes\varphi \vdash \boxtimes\psi$;

(ii) $\boxtimes\varphi \vdash \varphi$;

(iii) $\boxtimes\varphi \vdash \boxtimes \boxtimes \varphi$;

(iv) $\boxtimes(\varphi \longrightarrow \psi) \vdash \boxtimes\varphi \longrightarrow \boxtimes\psi$;

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(iv) $\boxtimes(\varphi \to \psi) \vdash \boxtimes\varphi \to \boxtimes\psi$;

(i) - (iii) follow from the fact that $\boxtimes$ is a comonad, as before.

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(i) If $\varphi \vdash \psi$ then $\boxtimes\varphi \vdash \boxtimes\psi$;

(ii) $\boxtimes\varphi \vdash \varphi$;

(iii) $\boxtimes\varphi \vdash \boxtimes \boxtimes \varphi$;

(iv) $\boxtimes(\varphi \rightarrow \psi) \vdash \boxtimes\varphi \rightarrow \boxtimes\psi$;

(iv) requires an argument that the meet of two invariant co-equations is again invariant. This is not difficult.

The invariance theorem, revisited

Lemma. $\langle A, \alpha \rangle \models \varphi$ *iff* $\langle A, \alpha \rangle \models \Box\varphi$.

The invariance theorem, revisited

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Lemma. $\langle A, \alpha \rangle \models \varphi$ *iff* $\langle A, \alpha \rangle \models \boxtimes\varphi$.

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Lemma. $\langle A, \alpha\rangle \models \varphi$ *iff* $\langle A, \alpha\rangle \models \boxtimes\varphi$.

Lemma. *Let* $[-]:\mathsf{Sub}_{\mathcal{E}}(UHC)\to\mathsf{Sub}_{\mathcal{E}_\Gamma}(HC)$ *be the right adjoint to* $U:\mathsf{Sub}_{\mathcal{E}_\Gamma}(HC)\to\mathsf{Sub}_{\mathcal{E}}(HC)$ *(so* $\Box = U\circ[-]$*). Then* $[\boxtimes\varphi]\models\varphi$.

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Lemma. $\langle A, \alpha\rangle \models \varphi$ *iff* $\langle A, \alpha\rangle \models \boxtimes\varphi$.

Lemma. $[\boxtimes\varphi] \models \varphi$.

Theorem. φ *is a generating coequation iff* $\varphi = \Box \boxtimes \varphi$.

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Theorem. φ *is a generating coequation iff* $\varphi = \Box \boxtimes \varphi$.

Theorem. $\Box \boxtimes \varphi \leq \boxtimes\Box\varphi$, *i.e., if* φ *is invariant, then so is* $\Box\varphi$.

The invariance theorem, revisited

Lemma. $\langle A, \alpha \rangle \models \varphi$ *iff* $\langle A, \alpha \rangle \models \Box\varphi$.

Lemma. $\langle A, \alpha \rangle \models \varphi$ *iff* $\langle A, \alpha \rangle \models \boxtimes\varphi$.

Lemma. $[\boxtimes\varphi] \models \varphi$.

Theorem. *φ is a generating coequation iff $\varphi = \Box \boxtimes \varphi$.*

Theorem. *$\Box \boxtimes \varphi \leq \boxtimes\Box\varphi$, i.e., if φ is invariant, then so is $\Box\varphi$.*

Theorem. *If Γ preserves non-empty intersections, then $\Box \boxtimes \varphi = \boxtimes\Box\varphi$.*

Some open questions

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$$\mathbf{V}_{\Box\varphi} = \mathbf{V}_{\varphi}$$

$$\mathbf{V}_{\boxtimes\varphi} = \mathbf{V}_{\varphi}$$

$$\mathbf{V}_{\varphi\wedge\psi} = \mathbf{V}_{\varphi} \cap \mathbf{V}_{\psi}$$

$$\mathbf{V}_{\exists_p\varphi} = ?$$

$$\mathbf{V}_{\neg\varphi} = ?$$

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- Is the preservation of non-empty intersections really relevant to the conclusion that $\Box\boxtimes = \boxtimes\Box$?
- What is the relation between the construction of a coequation φ and the corresponding covariety?
- What applications do these “non-behavioral” covarieties have in computer programming semantics?